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Multiperiod and Online Optimization for Fleet Defense: Centralized and Distributed Approaches

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In this paper, we consider several extensions of the asset-based weapon-to-target assignment problem whose objective is to protect assets in a fleet from incoming threats. We prove that this highly nonlinear mixed-integer optimization (MIO) problem can be efficiently solved with lazy constraints techniques, and therefore optimal solutions can be obtained online for instances of practical size. This paper also introduces a new extended MIO formulation for multiperiod scenarios, when the fleet has to plan against several consecutive attacks. Finally, we develop communication and coordination protocols for the decentralized version of the problem, in which captains of the assets have to make local decisions based on their own objectives and some limited communication with other ships. The suggested protocol uses robust optimization principles and generates weapon assignments that significantly improve upon a no-communication decentralized solution.

Key words: OR in defense, weapon-target assignment, multi-agent optimization, lazy constraints

1. Introduction

In this paper, we consider generalizations of the asset-based weapon-to-target assignment (WTA) problem, first introduced by Manne (1958), which is one of the most fundamental defense-related applications of operations research. We model the situation when a set of ships that have access to various weapons need to decide how to deploy their weapons optimally in order to protect themselves against a variety of incoming threats. Each ship has a limited number of kinetic (hard-kill) and non-kinetic (soft-kill) weapons. The hard-kill weapons aims at physically destroying an incoming missile, while soft-kill weapons use electronics to deflect threats from ships, for instance, by jamming or creating an imaginary target. The use of soft-kill weapons provides flexibility but adds complexity to the allocation of weapons to targets since such weapon not only can simultaneously affect multiple threats, but can also adversely deflect a missile from one ship towards another one.

A fleet defense optimization problem has two main objectives. The first one is to maximize the assets survival probabilities, and the second one is to minimize the resource used. The second objective is important when there are potential attacks over several periods, and there is a need to conserve weapons to address future raids of incoming threats.

1.1. Previous research

The single period weapon resource coordination problem has been studied before. In most of the cases researchers addressed the problem of coordinating only hard-kill weapons, which is known as the classical WTA problem (Manne 1958). It assumes that each assignment of weapons to target has its own efficacy, and the decision maker tries to find an optimal allocation of available weapons against incoming threats. Existing formulations of the WTA problem include defense of a single ship as well as of multiple assets.

The formulation of the single ship defense problem can be written as an integer optimization problem with convex logarithmic objective (Ahuja et al. 2007). This problem is nonlinear since several weapons may target the same threat. In this case, the probability of survival of the threat decreases proportionally to the *product* of the efficacies of the assigned weapons. The major drawback of this formulation is that it is not computationally tractable for even small instances with 20 weapons and 20 threats. This is why Ahuja et al. (2007) proposed network flow based heuristics. An alternative heuristic to the single ship defense problem involving enumeration of all possible assignments of multiple weapons is due to Bogdanowicz and Coleman (2008b).

Another popular approach to address the single ship WTA problem is to assume that each threat can only be engaged by at most one weapon system and therefore eliminate non-linearity in the formulation. Under this simplifying assumption, various algorithms have been proposed based on auction theory (Cheung and Chung 2010, Bertsekas 1990) or elaborate greedy methods (Bogdanowicz and Coleman 2008a).

The multi-asset WTA problem with hard-kill weapons is even more complex and nonlinear than the single-asset one since one threat may simultaneously target more than one ship. Hosein and Athans (1990) have suggested a heuristic method assuming that all weapons have the same efficacy, while other approximating algorithms without this assumption based on simulated annealing techniques have been presented in (Malhotra and Jain 2001, Bisht 2004). There are also methods based on network flow and auction theory that address the multi-asset WTA problem under the assumption that one threat can be engaged by at most one available hard-kill weapon (Bogdanowicz and Coleman 2008a, 2007). Murphey (2000) and Cai et al. (2006) provide comprehensive surveys of different variants of the WTA problem.

Becker et al. (2013) recently obtained some important results on the WTA problem with multiple ships in a fleet. First, they modeled the exact centralized WTA problem with both hard-kill and soft-kill weapons in a form of a binary second-order cone problem (SOCP) (Boyd and Vandenberghe 2004). Optimization problems of this type can be efficiently solved for offline purposes by commercial solvers for instances of typical size (Gurobi Optimization Inc. 2016). The authors also developed a heuristic based on linearization of second-order cone formulation that is fast enough for real-time applications. In this case, the optimal solution of the SOCP serves as a benchmark for the online heuristic. Finally, they considered a decentralized setting, for which they adapted their heuristic method and introduced a message passing algorithm. Their decomposition approach allows each ship to solve its local defense subproblem with updated data from other assets. However, the total amount of data that needs to be broadcast among ships is relatively large for such an online decision making, and therefore this protocol cannot be applied in many real world situations.

1.2. Contributions

In what follows, we denote by the fleet defense problem (FDP) any extension of the classical WTA problem that includes at least one additional feature: resource minimization objective, both hard-kill and soft-kill weapons, multiple consecutive raids or communication between agents in the decentralized setting. This paper generalizes and improves some of the results on the FDP introduced in (Becker et al. 2013). Our main contributions are as follows.

1. In Section 2, we develop an efficient algorithm solving the exact SOCP formulation of multi-asset FDP with hard-kill and soft-kill weapons in real time for instances of practical size. Our approach uses lazy constraints techniques applied to the highly nonlinear mixed-binary problems.

2. We propose in Section 3 a new method to address the multiperiod FDP introducing binary adaptive survival indicators and leveraging modeling power of MIO. The new multiperiod formulation generates more efficient weapon assignments in case of a sequence of attacks and remains computationally tractable.

3. We extend in Section 4 the FDP to a decentralized setting under uncertainty using a Robust Optimization (RO) approach. In this setting, each ship optimizes by itself with some limited communication with other ships and limited information on the severity of the threats to its companions and on the efficacy of their weapons.

4. We demonstrate in Section 5 using extensive computational experiments that both single and multiperiod formulations of the FDP can be solved online for instances of typical size. We also show the effectiveness of the suggested RO-based communication and cooperation protocol in comparison with a completely decentralized setting with no communication among ships and the idealistic centralized coordination.

The lazy constraints methodology and the approach for modeling multiple periods (that we develop in Sections 2 and 2, respectively) may be useful for addressing other challenging problems, such as transportation problems with nonlinear system effects (Manne 1958) and constrained resource allocation problems (Leboucher et al. 2013). The way we introduce communication decision variables (Section 4) may also be useful to model cooperation between agents in multi-agent optimization frameworks as developed in (Terelius et al. 2011) and (Lobel et al. 2011).

2. An Optimization Model for the Single Period FDP

In this section, we describe a nonlinear MIO formulation of the single period FDP that incorporates both hard-kill and soft-kill weapons (Becker et al. 2013). We also develop a new lazy constraints algorithm that efficiently solves this nonlinear problem by leveraging separating hyperplanes for polynomial constraints.

2.1. A Centralized Mathematical Formulation

Following Becker et al. (2013), we assume in this paper that all threats have a lethality value of 1, that is a ship is destroyed if it is hit by any given threat. In a centralized setting, we also presume that all ships in the fleet have common knowledge about where the incoming threats are headed, as well as assets can accurately estimate their weapons efficacy against these threats. All necessary notation is consolidated in Table 1.

The fleet consists of S ships and needs to protect itself from the set of incoming threats T. Each ship s in the fleet has two sets of weapons: hard-kill (denoted by H_s) and soft-kill (K_s). The binary decision variable x_{ij} is equal to 1, if weapon $i \in W = \bigcup_{s\in S}(H_s \cup K_s)$ is to against the threat $j \in T$. An assignment of weapon i to any given threat incurs firing cost r_i. There is a threat targeting probability matrix $\mathbf{Q} = \{Q_{js}\}$ for $j \in T$, $s \in S$, as well as hard-kill weapon efficiency matrix $\mathbf{P} = \{P_{ij}\}$ for $i \in H$, $j \in T$.

We assume that an assignment of a soft-kill weapon $i \in K = \bigcup_{s\in S} K_s$ to threat $k \in T$ may affect other threats, and introduce in this case new threat targeting probabilities $\mathbf{R} = \{R(j, s \,|\, i, k)\}$ for $j \in T$, $s \in S$, $i \in K$, $k \in T$. For all soft-kill weapons $i \in K$ and threats $j \in T$, set A_{ij} consists of all threats affected, if weapon i is assigned to threat j. We denote the set of all soft-kill assignments $(i, j) \in K \times T$ that can affect threat $k \in T$ as $B_k = \{(i, j) : k \in A_{ij}\}$.

In this formulation, we do not allow any given threat j to be affected by more than one soft-kill weapon. In order to model this, we introduce auxiliary binary variable y_j equal to 1, if threat j is not affected by any of soft-kill weapons. We also denote the probability that threat j hits an asset s by $D_{js} \in [0, 1]$, for $j \in T$, $s \in S$.

Each ship $s \in S$ in the fleet has a couple of predefined and globally known parameters: α_s is an asset priority that reflects the importance of the ship, and γ_s is a probabilistic survival threshold. We assume that ship s does not need additional protection if its survival probability is at least γ_s.

We introduce a continuous multiplicative slack variable w_s that indicates by how much the survival probability of ship s is below its survival threshold. If ship s is protected with probability at least γ_s, then $w_s = 1$, and w_s is strictly greater than one, otherwise (see Eq. (1c) below). The latter case implies a penalty in the objective function (see Eq. (1a) below). If the penalty λ is large enough, then the decision maker always prefers to fire an additional missile (if there is one available) to reduce a slack w_s, rather than to save this weapon.

Having introduced necessary notations, we formulate the centralized multi-asset single period FDP in the form of nonlinear mixed-binary optimization problem as follows:

$$\min_{\substack{\mathbf{x},\mathbf{y},\\ \mathbf{w},\mathbf{D}}} \quad \sum_{i\in W}\sum_{j\in T} r_i\, x_{ij} + \lambda \cdot \sum_{s\in S} \alpha_s (w_s - 1) \tag{1a}$$

$$\text{s.t.} \quad 1 \le D_{js}\left((Q_{js})^{-1} \cdot y_j + \sum_{(i,k)\in B_j} \big(R(j,s\,|\,i,k)\big)^{-1} \cdot x_{ik} \right) \prod_{i\in H}\left(1 + \frac{P_{ij}\, x_{ij}}{1-P_{ij}}\right), \quad \forall s\in S, j\in T \tag{1b}$$

$$\gamma_s \le w_s \prod_{j\in T}(1 - D_{js}), \quad \forall s \in S \tag{1c}$$

$$\sum_{j\in T} x_{ij} \le 1, \quad \forall i \in W \tag{1d}$$

$$y_j + \sum_{(i,k)\in B_j} x_{ik} = 1, \quad \forall j \in T \tag{1e}$$

$$1 \le w_s, \quad \forall s \in S \tag{1f}$$

$$0 \le D_{js} \le 1, \quad \forall s\in S,\, j\in T \tag{1g}$$

$$x_{ij},\, y_j \in \{0,1\}, \quad \forall i\in W,\, j\in T. \tag{1h}$$

The objective function (1a) consists of two parts. The first double sum is equal to the total weapon firing cost, while the second one represents the total penalty in case some ships have survival probabilities below predefined thresholds. If problem data admit a solution with all ships having survival probabilities greater than or equal to their corresponding survival thresholds, then all auxiliary multiplicative slacks $w_s = 1$ for all $s \in S$, and the cost function of the FDP (1a) is to minimize total weapons used. In a stressed scenario, when it is impossible to protect all ships in the fleet with probabilities above their survival thresholds, the cost function models a preferential defense strategy trying to maximize survival probabilities of the most important ships of the fleet with higher priority ranks α_s.

Given that higher priority ships have larger values of parameter α_s, the optimizer always prefers to protect important assets to their survivability thresholds before less important ones.

The probability D_{js} that threat j will leak through the defense and destroy ship s is given by:

$$D_{js} = \left\{ (Q_{js})^{y_j} \prod_{(i,k)\in B_j} \Big(R(j,s\,|\,i,k) \Big)^{x_{ik}} \right\} \prod_{i\in H} (1-P_{ij})^{x_{ij}}, \tag{2}$$

where Q_{js} denotes the initial probability that threat j targets ship s. If there are no soft-kill interactions with threat j, that is $y_j = 1$, then the second product in brackets

$$\prod_{(i,k)\in B_j} \Big(R(j,s\,|\,i,k) \Big)^{x_{ik}} = 1,$$

since $x_{ik} = 0$ for all $(i,k) \in B_j$ according to Eq. (1e), and therefore the targeting probability Q_{js} remains unchanged. But the initial targeting probability Q_{js} can be substituted with another value $R(j,s\,|\,i,k)$, if soft-kill weapon $i \in K$ is fired against threat $k \in T$, that is x_{ik}=1, and this interaction (i,k) is in a set of soft-kill assignments B_j that affect threat j. The last product

$$\prod_{i\in H} (1-P_{ij})^{x_{ij}}$$

in the right hand side of (2) defines the probability that threat j is not destroyed by any of the independent hard-kill weapons $i \in H$. Given that all decision variables x_{ij} and y_j in Eq. (2) are binary, an alternative way to the define destruction probability D_{js} is as follows:

$$D_{js} = \left\{ (Q_{js})^{-1} y_j + \sum_{(i,k)\in B_j} \Big(R(j,s\,|\,i,k) \Big)^{-1} x_{ik} \right\}^{-1} \cdot \prod_{i\in H} \Big(1 + \frac{P_{ij}\, x_{ij}}{1-P_{ij}} \Big)^{-1}. \tag{3}$$

In order to be able to invert elements of matrices $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ in Eq. (3), we assume that all probabilities are between ε and $1-\varepsilon$, for some small predefined value $\varepsilon > 0$. Thus, Ineq. (1b) expresses that the probability D_{js} that threat j destroys a ship s is bounded below by the probability that j is attacking asset s and it is not ruined by all hard-kill weapons fired against j.

Since the lethality value of all threats equals one, the probability that ship $s \in S$ survives in a raid is equal to the probability that ship is not damaged by all independent threats:

$$\prod_{j \in T} (1 - D_{js}). \tag{4}$$

Given that the auxiliary multiplicative slack w_s is heavily penalized in the cost function (1a), we will always have $w_s = 1$ in an optimal solution for ship s, whose survival probability (4) is at least γ_s. Hence, Ineq. (1c) states the survivability threshold for ship s.

We express the requirement that each weapon can be engaged no more than once by Ineq. (1d). Equation (1e) states that each threat $j \in T$ can be affected by at most one soft-kill weapon. The last group of constraints (1f) – (1h) defines upper and lower bounds or imposes integrality for decision variables of the problem.

The only nonlinear constraints of the problem (1) are (1b) and (1c). Both of them can be easily reformulated in the following form:

$$1 \le \prod_{i \in \mathcal{I}} u_i, \qquad \text{where variables } u_i \text{ are non-negative, and } \mathcal{I} \text{ is some finite index set.} \tag{5}$$

Becker et al. (2013) proved that inequality of the form (5) defines a convex set and has a representation as a projection of a system of rotated second-order cone constratints. Hence, the FDP (1) can be solved offline by commercial solvers (for instance, Gurobi Optimization Inc. (2016)) as a convex quadratically-constrained binary optimization problem (Boyd and Vandenberghe 2004) within minutes depending on the instance size. While this approach yields an optimal solution to the FDP that can serve as a benchmark, it cannot be used in real time.

2.2. Lazy Constraints Algorithm for the FDP

In this subsection, we propose a new efficient algorithm for solving problem (1) leveraging the special structure of nonlinear inequalities (1b), (1c) and MIO lazy constraints techniques (IBM Knowledge Center 2017).

Algorithm 1 LAZY method for the FDP

procedure LAZY($\mathbf{Q}, \mathbf{P}, \mathbf{R}, \boldsymbol{\alpha}, \boldsymbol{\gamma}$)
    Predefine computational tolerance $\delta > 0$.
    Solve relaxation of (1) with the omitted nonlinear constrains (1b), (1c).
    Denote the feasible set of the relaxed problem by $\mathcal{P}$ and its optimal solution by $\mathbf{X} = (\mathbf{x}^*, \mathbf{y}^*, \mathbf{w}^*, \mathbf{D}^*)$.
    while solution $\mathbf{X}$ does not satisfy constraints (1b), (1c) with tolerance δ **do**
        Find a hyperplane π separating the current solution $\mathbf{X}$ from the feasible set $\mathcal{P}$.
        Update set $\mathcal{P}$ by adding the separating inequality that represents π.
        Re-solve the relaxation of problem (1) with the augmented $\mathcal{P}$.
        Update the optimal solution $\mathbf{X}$.
    return $\mathbf{x}^*$, i.e., assignments of weapons to threats.

In order to implement LAZY algorithm, we next derive in closed form the separating hyperplanes associated with nonlinear constraints (1b), (1c) of the form (5).

LEMMA 1. *Let us consider a convex set* $\Sigma = \left\{\mathbf{x} \in \mathbb{R}^n_+ \,|\, \gamma \le \prod_{i=1}^n x_i\right\}$ *for some* $\gamma > 0$, *where* $\mathbb{R}^n_+ = \{\mathbf{x} \,|\, x_i > 0,\, i = 1, \ldots, n\}$. *Then any point* $\mathbf{y} \in \mathbb{R}^n_+ \setminus \Sigma$ *can be separated from set* Σ *by the separating hyperplane*

$$\sum_{i=1}^n \frac{x_i}{\hat{y}_i} = n, \tag{6}$$

where point $\hat{\mathbf{y}} = (\hat{y}_1, \ldots, \hat{y}_n) \in \mathbb{R}^n_+ \cap \Sigma$ *is defined in one of the following ways:*

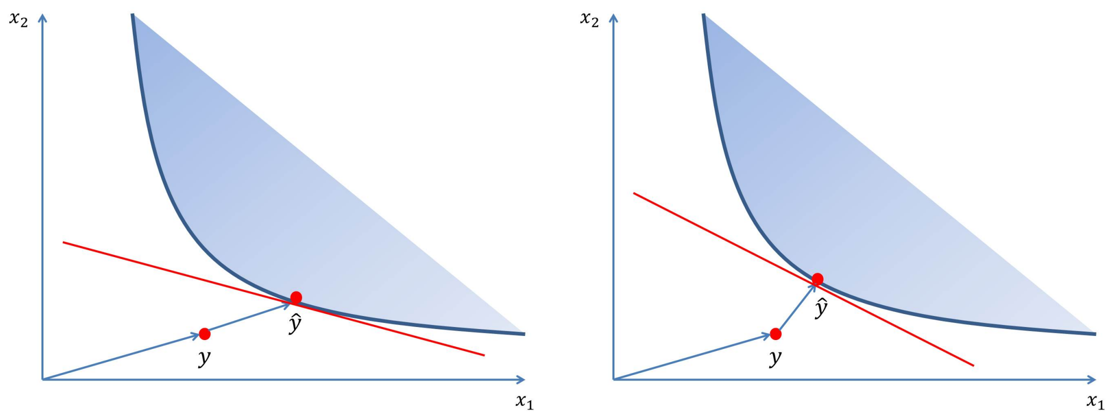
$$1) \quad \hat{y}_i = y_i t^*, \ \textit{ for } \ i = 1, \ldots, n \ \textit{ and } t^* = \sqrt[n]{\frac{\gamma}{y_1 y_2 \ldots y_n}}; \textit{ or} \tag{7a}$$

$$2) \quad \hat{y}_i = \frac{1}{2}\left(y_i + \sqrt{y_i^2 - 4\mu^*\gamma}\right), \qquad \textit{for } i = 1, \ldots, n, \ \textit{and } \mu^* \textit{ is the unique root} \tag{7b}$$

of the equation $g(\mu) = 0$, *where*

$$g(\mu) = \prod_{i=1}^n \left(\frac{y_i + \sqrt{y_i^2 - 4\mu\gamma}}{2}\right) - \gamma. \tag{8}$$

The first definition of point $\hat{\mathbf{y}}$ *(7a) characterizes linear projection of point* $\mathbf{y}$ *onto the boundary of a convex set* Σ*, while the second one (7b) defines an orthogonal projection (Fig. 1).*



(a) Linear projection of point $\mathbf{y}$ (b) Orthogonal projection of point $\mathbf{y}$

Figure 1 Separating hyperplanes for set Σ.

The proof of Lemma 1 can be found in the electronic companion of the paper.

3. An Optimization Model for the Multiperiod FDP

This section develops a nonlinear MIO formulation of the FDP for the case of multiple consecutive attacks. The extended multiperiod setting has to address a tradeoff between securing the survival probabilities above the thresholds in the immediate attack and weapon conservation objectives for future attacks. We also model a possibility that some of the ships of the fleet can be destroyed in the current raid, and therefore in consecutive attacks there is no need to protect them and we cannot use their weapons.

Similar to the single period setting, each of the N attacks (indexed by $\tau = 1, \ldots, N$) consists of a set of threats T^τ. We denote the probability that at time-step τ threat $j \in T^\tau$ targets ship $s \in S$ as Q^τ_{js}. The corresponding weapon efficacy matrices are represented by $\mathbf{P}^\tau = \{P^\tau_{ij}\}$ for all raids $\tau = 1, \ldots, N$, weapons $i \in W$ and threats $j \in T^\tau$. The possible changes in threat targeting probabilities induced by the implementation of soft-kill weapons are described by matrices $\mathbf{R}^\tau = \{R^\tau(j, s\,|i, k)\}$. Finally, sets of soft-kill interactions B^τ_k that affect threat $k \in T^\tau$ are defined as in Section 2.1 for all time-steps $\tau = 1, \ldots, N$.

The global objective of the multiperiod FDP is to protect its ships in all attacks in the most resource-efficient way, that is minimizing weapons used. If the ideal scenario when all ships are protected with survival probabilities above predefined thresholds is infeasible, we implement preferential defense strategy that ensures that the most valuable assets are defended with a higher priority. In order to model this strategy, we use the same decision variables $x_{ij}^\tau, y_j^\tau, w_s^\tau, D_{js}^\tau$ as in single period formulation (1) with the only difference that now they have an additional time-index $\tau = 1, \ldots, N$. The only new variables that we need for the multiperiod setting are binary destruction indicators u_s^τ that are equal to 1, if ship s is destroyed before the beginning of attack $\tau = 1, \ldots, N$. Thus, the multiperiod FDP is given by a similar to (1) formulation:

$$\min_{\substack{\mathbf{x},\mathbf{y},\mathbf{w}\\ \mathbf{u},\mathbf{D}}} \quad \sum_{\tau=1}^{N} \left(\sum_{i\in W} \sum_{j \in T^\tau} r_i\, x_{ij}^\tau + \lambda \cdot \sum_{s\in S} \alpha_s (w_s^\tau - 1) \right) \tag{9a}$$

$$\text{s.t.} \quad 1 \le D_{js}^\tau \left((Q_{js}^\tau)^{-1} \cdot y_j^\tau + \sum_{(i,k)\in B_j^\tau} \big(R^\tau(j,s\,|\,i,k)\big)^{-1} \cdot x_{ik}^\tau \right) \prod_{i\in H} \Big(1 + \frac{P_{ij}^\tau}{1-P_{ij}^\tau}\, x_{ij}^\tau\Big), \quad \forall \tau = 1,\ldots,N;\ s\in S,\ j\in T^\tau \tag{9b}$$

$$\gamma_s \le \big(w_s^\tau + M u_s^\tau\big) \prod_{j\in T^\tau} (1 - D_{js}^\tau), \quad \forall \tau = 1,\ldots,N;\ s\in S \tag{9c}$$

$$\sum_{\tau=1}^{N} \sum_{j\in T^\tau} x_{ij}^\tau \le 1, \quad \forall i \in W \tag{9d}$$

$$y_j^\tau + \sum_{(i,k)\in B_j^\tau} x_{ik}^\tau = 1, \quad \forall \tau = 1,\ldots,N;\ j\in T^\tau \tag{9e}$$

$$u_s^1 = 0, \quad \forall s \in S \tag{9f}$$

$$\frac{w_s^\tau - 1}{M} \le u_s^{\tau+1} \le M(w_s^\tau - 1), \quad \forall \tau = 1,\ldots,N-1;\ s\in S \tag{9g}$$

$$u_s^\tau \le u_s^{\tau+1}, \quad \forall \tau = 1,\ldots,N-1;\ s\in S \tag{9h}$$

$$x_{ij}^\tau \le 1 - u_s^\tau, \quad \forall \tau = 1,\ldots N;\ s\in S,\ i\in H_s\cup K_s,\ j\in T^\tau \tag{9i}$$

$$1 \le w_s^\tau, \quad \forall \tau = 1,\ldots,N;\ s\in S \tag{9j}$$

$$0 \le D_{js}^\tau \le 1, \quad \forall \tau = 1,\ldots,N;\ s\in S,\ j\in T \tag{9k}$$

$$x_{ij}^\tau,\ y_j^\tau, u_s^\tau \in \{0,1\}, \quad \forall \tau = 1,\ldots,N;\ i\in W,\ j\in T,\ s\in S. \tag{9l}$$

The objective function (9a) models the preferential defense strategy, while each constraint of type (9b) defines a lower bound on the probability $D^{\tau}_{j,s}$ that threat $j \in T^{\tau}$ ruins ship s at time-step τ. Similarly as before, Ineq. (9c) states the survivability threshold for asset s at time-step τ. The only difference is that now we substitute multiplicative slack w^{τ}_s with a sum

$$w^{\tau}_s + M u^{\tau}_s, \tag{10}$$

where M is a large enough number. If ship s is not destroyed at the beginning of time-step τ and we need to protect it, that is $u^{\tau}_s = 0$, then factor (10) does not change and equals w_s as before (1). At the same time, if ship s was destroyed in one of the preceding raids, that is $u^{\tau}_s = 1$, then Ineq. (9c) does not impose any constraints on the remaining decision variables $\mathbf{w}, \mathbf{D}$ and is automatically satisfied. It means that we do not need to defend already destroyed ships.

Inequalities (9d) express that we cannot fire a weapon more than once, while equations of type (9e) state that at any given time-step $\tau = 1, \ldots, N$ each threat $j \in T^{\tau}$ can be affected by at most one soft-kill weapon to avoid interdependencies.

Constraints (9f) – (9h) define destruction indicators u^{τ}_s. In the beginning of the first raid $\tau = 1$ all ships in the fleet are operating (9f). If at time-step τ ship s is protected with probability at least γ_s, then $w_s = 1$ and $u^{\tau+1}_s = 0$ (9g). At the same time, if $w^{\tau}_s > 1$, that is, ship s is not defended with probability at least γ_s at time-step τ, then Inequalities (9g) force the destruction indicator $u^{\tau+1}_s$ to switch its value to 1. In this case, we consider asset s to be destroyed in all consecutive raids $\tau + 1, \ldots, N$, which is represented by the set of constraints (9h).

Inequalities (9i) prohibit to use weapons of destroyed assets. The last group of constraints (9j) – (9l) defines upper and lower bounds or imposes integrality for decision variables of the problem.

The decision maker should solve multistage optimization problem (9) updating uncertain parameters in the end of each attack. More precisely, binary survival indicators u^{τ}_s for $s \in S$ and $\tau = 1, \ldots, N$ are based on estimated survival probabilities (9c). At the same time, it is possible that ship s will be destroyed even if its survival probability is higher than the corresponding threshold γ_s, and vice versa. On the other hand, in the end of each attack τ a true ship survivability vector $\hat{\mathbf{u}}^\tau = (\hat{u}_1^\tau, \ldots, \hat{u}_S^\tau)$ becomes available for the decision maker, that allows her to substitute an estimated vector $\mathbf{u}^\tau$ with the accurate one $\hat{\mathbf{u}}^\tau$.

The only nonlinear constraints (9b) and (9c) in optimization problem (9) again have the form of (5). This is why it is possible to solve the multiperiod FDP as a convex mixed-binary SOCP. Moreover, we will demonstrate in Section 5 that it is possible to solve instances of (9) of typical size in real time leveraging lazy constraints algorithm designed in Section 2.2.

4. Decentralized Approach for the Fleet Defense Problem

The optimization problems introduced in Sections 2 and 3 are designed for the setting when all data about threats and weapons are globally known to all ships. However, in practice it is not always the case since each ship possesses some local information. Furthermore, there is usually not enough time to broadcast all the local data to other ships, allow everyone to solve the same centralized optimization problem and execute the local part of the global optimal solution. This is why designing a distributed approach is important for practical applications of the FDP. In this setting, solving the global optimization problem is similar to many multi-agent communication and coordination problems, for instance, (Ren and Sorensen 2008, Sariel and Balch 2006, Maza et al. 2011), when several independent players cooperate in order to achieve a common goal.

In this section, we introduce a decentralized algorithm for solving the FDP leveraging and improving ideas formulated in (Becker et al. 2013). The authors suggested an iterative cooperation protocol utilizing so-called leakage and linkage matrices that should be broadcast. These messages contain data about local weapon effectiveness and assignments and allow other ships to update their local decisions in order to improve fleet-wide objectives. The main disadvantage of the existing communication protocol (ECP) from (Becker et al. 2013) is that the total size of the broadcast messages can be prohibitively large for practical applications. A brief comparison of the ECP and the new communication protocol (NCP) that we present here is summarized in Section 4.2.

Our new approach for distributed operations and communication is optimization-based. Namely, we do not broadcast all information about the field of action. In contrast, we determine essential fragments of information that should be shared among ships as an optimal solution of the specific two-stage RO problem (see Eq. (11) and Eq. (14) below). In this section, we design the new protocol for the case of a single attack ($N = 1$) and the presence of hard-kill weapons only ($K = \emptyset$).

4.1. The Cooperation Protocol

Following (Becker et al. 2013), we assume that all data of the FDP described in Section 2 are split into globally known and individually known blocks. On one hand, every ship s in the fleet possesses the following global data:

- Initial targeting probabilities: $Q_{js'}, \quad \forall s' \in S, \quad j \in T$
- Probabilistic survival thresholds: $\gamma_{s'}, \quad \forall s' \in S$
- Asset priorities: $\alpha_{s'}, \quad \forall s' \in S$
- Predefined minimum accuracy of hard-kill interceptors: $\delta_{s'}, \quad \forall s' \in S$.

On the other hand, each ship s has some private data that are not available to other ships:

- Efficiency of the weapons against all threats: $P_{ij}, \quad \forall i \in H_s, \quad j \in T$
- Cost of own weapons: $r_i, \quad \forall i \in H_s$.

Moreover, when ship s makes some local (possibly tentative) weapon assignment decisions, they are also not available for the rest of the fleet unless explicitly broadcast. Having defined two types of data, the architecture of cooperation among ships under the assumption of limited time for communication can be outlined as three consecutive steps:

1. Each ship independently solves an optimization problem that simultaneously generates an initial self-defense strategy and determines which information about local weapon assignments should be broadcast.
2. Ships broadcast messages across the fleet.
3. Ships update their initial solutions based on revealed information from companions.

The details of each of the steps including explicit optimization formulations to be solved, connections between weapon assignments and messages to broadcast as well as updating rules for targeting probabilities are discussed below.

Step 1: Initial Solution

In this step, each ship calculates a preliminary solution that employs globally known data and local information about its own weapons. We propose an optimization-based variant of (1) for the initial solution that realizes a self-defense strategy. In this model, each ship tries to maximize the survival probabilities across the fleet with a much higher emphasis on defending itself. On top of this, each ship simultaneously tries to identify which information about its weapon assignments would be the most useful to know for companions and therefore should be broadcast.

This decentralized optimization problem for a fixed ship $s \in S$ can be represented as follows:

$$z_1(s) = \min_{\substack{\mathbf{x},\mathbf{w},\\ \mathbf{f},\mathbf{D}}} \quad \sum_{i\in H_s}\sum_{j\in T} r_i x_{ij} + \lambda \cdot \alpha_s(w_s - 1) + \mu \cdot \sum_{s'\in S\backslash\{s\}} \alpha_{s'}(w_{s'} - 1) \tag{11a}$$

$$\text{s.t.} \quad 1 \le (Q_{js})^{-1} D_{js} \prod_{i\in H_s}\Big(1 + \frac{P_{ij}\, x_{ij}}{1-P_{ij}}\Big), \quad \forall j \in T_s \tag{11b}$$

$$D_{js'} = Q_{js'}(1 - \delta_s f_{js}), \quad \forall s' \in S\backslash\{s\}, j \in T_{s'} \tag{11c}$$

$$\sum_{j\in T} f_{js} \le \hat{F}_s \tag{11d}$$

$$\sum_{i\in H_s} x_{ij} \ge f_{js}, \quad j \in T \tag{11e}$$

$$\gamma_{s'} \le w_{s'} \prod_{j\in T}(1 - D_{js'}), \quad \forall s' \in S \tag{11f}$$

$$\sum_{j\in T} x_{ij} \le 1, \quad \forall i \in H_s \tag{11g}$$

$$1 \le w_{s'}, \quad \forall s' \in S \tag{11h}$$

$$0 \le D_{js'} \le 1, \quad \forall s' \in S, j \in T \tag{11i}$$

$$x_{ij}, f_{js} \in \{0,1\}, \quad \forall i \in H_s, j \in T. \tag{11j}$$

Optimization problem (11) employs new binary decision variables $\mathbf{f}$: we set variable f_{js} to be equal to 1, if a ship $s \in S$ decides to broadcast a message that it fires at least one weapon at threat $j \in T$. It is important to note that, due to communication constraints, ship s does not specify neither the total amount of weapons fired at j, nor the efficiency of the assigned weapons.

The objective function (11a) is similar to the objective from the centralized case (1a) with the only difference that now the importance of saving the ship s (measured by penalty factor λ) is much higher than the importance of protecting other companions (represented by a factor μ). Inequalities (11b) define destruction probabilities D_{js} for the ship s as before (1b), conditional on the presence of hard-kill weapons only.

On the other hand, we define destruction probabilities $D_{js'}$ for the rest of the ships $s' \neq s$ in a new way (11c). The key idea is that another ship s' knows that ship s fires at least one weapon at threat $j \in T_{s'}$ only if ship s notifies everyone about it, which corresponds to the case $f_{js} = 1$. In this scenario, ship s' realizes that the initial targeting probability $Q_{js'}$ can be decreased. More precisely, it can be multiplied by $(1 - \delta_s)$, where δ_s is a globally known minimum accuracy of weapons of ship s. Thus, from the perspective of ship s, Eq. (11c) illustrates by how much ship s will help ship s', if it fires at least one weapon against threat $j \in T_{s'}$ and broadcasts a corresponding message.

Inequality (11d) states that the total number of messages that ship s can broadcast is at most $\hat{F}_s$, while constraint (11e) makes weapon assignments $\mathbf{x}$ and communication decisions $\mathbf{f}$ consistent with each other. The rest of the constraints are identical to the centralized formulation (1).

It is possible to solve the decentralized version of the FDP (11) with $K_s = \emptyset$ by lazy constraints techniques almost instantly (see Section 5). This formulation is computationally easier than the centralized version since there are only hard-kill weapons and a large number of nonlinear constraints of type (11b) are substituted with linear equations (11c). Since these computations are organized independently for all assets $s \in S$, then the initial solutions become available within a matter of seconds after a set of threats $T = \{T_s : s \in S\}$ is detected.

It is worth mentioning that the fully decentralized scenario of the FDP without communication when ships independently try to protect only themselves also has the form of optimization problem (11). In this case, it is sufficient to assume zero communication capacities $\hat{F}_s = 0$ for all $s \in S$ and zero penalty $\nu = 0$ for violations of survival of other ships.

Step 2: Broadcasting Messages

At the second stage of the protocol, ships broadcast information about their initial weapon assignments according to the optimal values of communication variables $\mathbf{f}^* = \{f^*_{js} \,|\, j \in T, s \in S\}$ derived by the solutions of (11).

Step 3: Updating Weapon Allocation

Based on the messages from Step 2, each ship s may update estimations of targeting probabilities $\mathbf{Q}$ since some of the threats become less dangerous. We introduce uncertain parameters $\tilde{Q}_{js'}(s)$, for $j \in T$, $s \in S$ and $s' \in S \backslash \{s\}$ representing a probability that threat j targets ship s' after the first stage of the protocol measured from the perspective of ship s.

Given that variables $\mathbf{f}$ contain only a portion of information about weapon assignments and the number of these messages is limited, for all ships in a fleet targeting probabilities $\tilde{\mathbf{Q}}$ remain uncertain. We can introduce the corresponding uncertainty sets as

$$U_{js'}(s) = \big\{\tilde{Q}_{js'}(s) \,|\, 0 \le \tilde{Q}_{js'}(s) \le \hat{Q}_{js'}(s)\big\}, \tag{12}$$

where upper bound $\hat{Q}_{js'}(s)$ is determined by globally available data:

$$\hat{Q}_{js'}(s) = Q_{js'} \prod_{s'' \neq s} (1 - f^*_{js''}\delta_{s''}). \tag{13}$$

Equation (13) states that from the perspective of ship $s \in S$ threat $j \in T$ targets ship $s' \in S$ with initial probability $Q_{js'}$ discounted by $(1 - f^*_{js''}\delta_{s''})$ for any ship s'' different from s that fires at least one weapon against this threat.

We denote by $\mathbf{U}(s)$ the cartesian product of sets defined in (12) that aggregate estimations of targeting probabilities $\tilde{Q}_{js'}(s)$ from the perspective of ship s as follows

$$\mathbf{U}(s) = \Big\{\big(\tilde{Q}_{11}(s), \ldots, \tilde{Q}_{TS}(s)\big) \,|\, \tilde{Q}_{js'}(s) \in U_{js'}(s) \text{ for } j \in T, s' \in S\Big\}.$$

In this step of the protocol, each ship s solves the FDP under uncertainty with respect to $\tilde{\mathbf{Q}}(s)$ incorporating additional information modeled by the set $\mathbf{U}(s)$ as follows:

$$z_3(s) = \min_{\mathbf{x},\mathbf{w},\mathbf{D}} \max_{\tilde{\mathbf{Q}}(s) \in \mathbf{U}(s)} \quad \sum_{i \in H_s} \sum_{j \in T} r_i \, x_{ij} + \lambda \cdot \alpha_s (w_s - 1) + \mu \cdot \sum_{s' \in S \backslash \{s\}} \alpha_{s'} (w_{s'} - 1) \tag{14a}$$

s.t.
$$1 \le \left(\tilde{Q}_{js'}(s)\right)^{-1} D_{js'} \prod_{i \in H_{s'}} \left(1 + \frac{P_{ij}\, x_{ij}}{1 - P_{ij}}\right), \quad \forall s' \in S, j \in T_{s'} \tag{14b}$$

$$\sum_{i \in H_s} x_{ij} \ge f^*_{js}, \quad j \in T \tag{14c}$$

$$\gamma_{s'} \le w_{s'} \prod_{j \in T} (1 - D_{js'}), \quad \forall s' \in S \tag{14d}$$

$$\sum_{j \in T} x_{ij} \le 1, \quad \forall i \in H_s \tag{14e}$$

$$1 \le w_{s'}, \quad \forall s' \in S \tag{14f}$$

$$0 \le D_{js'} \le 1, \quad \forall s' \in S, j \in T \tag{14g}$$

$$x_{ij} \in \{0,1\}, \quad \forall i \in H_s, j \in T. \tag{14h}$$

Problem (14) is a nonlinear robust optimization problem (Ben-Tal et al. 2009, Bertsimas et al. 2011). The objective function (14a) reflects that the decision maker controls only weapon assignment variables $\mathbf{x}$ (and auxiliary dependent variables $\mathbf{w}$ and $\mathbf{D}$), while targeting probabilities $\tilde{\mathbf{Q}}(s)$ are uncertain parameters.

Inequalities (14b) determine the destruction probabilities $D_{js'}$ as before with the only difference that now the first factor $\left(\tilde{Q}_{js'}(s)\right)^{-1}$ is not a constant, but rather an uncertain parameter. Given that each inequality (14b) should hold for all values of $\tilde{Q}_{js'}(s)$ in set $U_{js'}(s)$, it is easy to see that (14b) should remain feasible for the smallest possible value of $\left(\tilde{Q}_{js'}(s)\right)^{-1}$ or, equivalently, for the largest possible value of $\tilde{Q}_{js'}(s)$. This value is known to be equal to $\hat{Q}_{js'}(s)$ as defined in (13). Thus, the robust optimization problem (14) is equivalent to a regular optimization problem if all uncertain parameters $\tilde{Q}_{js'}(s)$ are substituted with the largest possible values $\hat{Q}_{js'}(s)$, that represent the most risk-averse scenario, and maximization with respect to $\tilde{\mathbf{Q}}(s)$ in (14a) is omitted. As such, formulation (14) can be efficiently solved by a combination of standard RO techniques (Bertsimas et al. 2011) and the LAZY algorithm from Section 2.2.

Inequalities (14c) state that if ship s announced at the first step of the communication protocol to allocate at least one weapon to threat $j \in T$ (i.e. $f^*_{js} = 1$), then it should stick to the broadcast decision. The rest of the constraints are identical to formulations (1) and (11).

The implementation of the robust optimization methodology for solving the FDP (14) delivers two major benefits. First, it allows us to model uncertainty with respect to targeting probabilities $\tilde{\mathbf{Q}}(s)$ remaining after communication at Step 2 by means of sets $\mathbf{U}(s)$. Second, consideration of the most risk-averse scenario represented by inner maximization in (14) is a reasonable choice given the danger and liability involved when solving the fleet defense problem.

4.2. Discussion

The main differences between the ECP from Becker et al. (2013) and the NCP designed in Section 4.1 can be summarized as follows:

1. The ECP is based on a linearized heuristic of the binary second-order cone optimization problem (1) aiming to make it fast enough for real world scenarios, while the NCP efficiently solves exact nonlinear problems (11) and (14) leveraging lazy constraints techniques.

2. The ECP involves an iterative process of alternation between local computations and message broadcasting until a stopping condition is met. In contrast, the NCP considers a more stressful scenario with sharp time limits for communication, and therefore it has exactly three steps.

3. The architecture of the ECP assumes that information that ships share among each other (leakage and linkage matrices) is predefined and always the same regardless of its importance. At the same time, a new protocol broadcasts not all data, but only its most useful pieces. Messages that ships transmit according to the NCP are defined as a part of an optimal solution of the optimization problem (11).

4. When the ECP is employed, then the total size of transmitted messages (consisting of several matrices) is fixed and may be too large for many practical situations. In contrast, the NCP allows the decision maker to control a number of messages that can be broadcast by tuning input parameters of the optimization problem (11) that represent communication capacities.

Another important advantage of the NCP designed in Section 4.1 is that its implementation can dramatically improve the global fleet objective

$$\mathrm{Z}_{total} = \sum_{s \in S} z(s), \quad \text{where} \tag{15}$$

$$z(s) = \sum_{i \in H_s} \sum_{j \in T} r_i \, x_{ij} + \lambda \cdot \alpha_s (w_s - 1) + \mu \cdot \sum_{s' \in S \setminus \{s\}} \alpha_{s'} (w_{s'} - 1). \tag{16}$$

The value of $z(s)$ represents the total weapon costs fired by ship s and a penalty if some of the ships in the fleet are not protected with required probabilities. We can compute this sum Z_{total} before and after implementation of the protocol. More precisely,

$$\mathrm{Z}_{total}^{start} = \sum_{s \in S} z_1(s) \quad \text{and} \quad \mathrm{Z}_{total}^{final} = \sum_{s \in S} z_3(s)$$

where $z_1(s)$ and $z_3(s)$ are the optimal solution values of (11) and (14), respectively.

The main idea of possible improvement is that the protocol aims to increase survival probabilities of insufficiently protected ships and thereby reduce heavily penalized slack variables w_s (16). In other words, it is not necessarily true that implementation of the protocol leads to a reduction of total weapon costs in comparison with the completely decentralized scenario, since some ships may spend additional weapons to protect insecure companions and therefore increase the total weapon expenses. At the same time, there is a dominating factor λ representing the paradigm that the fleet is much more concerned about survival of the ships than weapon costs. As a result, a solution with higher survival probabilities may result in a significant reduction of the objective value Z_{total}.

The cooperation protocol described in Section 4.1 may only decrease targeting probabilities for all ships in the fleet. Namely, Step 2 implies that $\hat{Q}_{js'}(s) \leq Q_{js'}$ for all ships $s, s' \in S$ and threats $j \in T$ according to (13). Thus, the main advantage of the suggested communication protocol is that survival probabilities of all ships may only increase and the objective values $z(s)$ of all ships may only decrease going from Step 1 to Step 3, which we prove formally in the following proposition.

PROPOSITION 1. *For any input data* $\mathbf{Q}, \mathbf{P}, \hat{\mathbf{F}}, \mathbf{r}, \boldsymbol{\alpha}, \boldsymbol{\delta}, \lambda, \mu$ *and any ship* $s \in S$ *the following inequality between objective values of optimization problems (11) and (14) holds:*

$$z_3(s) \leq z_1(s). \tag{17}$$

The proof of Proposition 1 can be found in the electronic companion of the paper. Combining result (17) from Proposition 1 and definition (15), we imply that $Z_{total}^{final} \leq Z_{total}^{start}$.

5. Computational Results

In this section, we empirically demonstrate the practical effectiveness of the MIO approach and lazy constraints techniques for both centralized and distributed counterparts of the FDP. Together with RO-based communication protocol, they allow the decision maker to obtain high-quality solutions of the FDP of typical size in real time.

5.1. Advantage of the MIO Approach

In this experiment, we present a simple heuristic algorithm that can be employed as a benchmark for solving the FDP. We empirically validate that the MIO approach to the FDP developed in Section 2 significantly outperforms this heuristic algorithm providing a better protection for the fleet and spending less weapons for this defense.

One possible heuristic method (that we denote by GREEDY) is to sort all incoming threats from the most dangerous to the fleet to the least dangerous, and then sequentially assign the most efficient weapons against these sorted threats. A rigorous definition of the algorithm is as follows.

Algorithm 2 GREEDY method for the FDP

procedure GREEDY($\mathbf{Q}, \mathbf{P}, \boldsymbol{\alpha}, \boldsymbol{\gamma}$)
  Set weapon assignment matrix $\mathbf{x} = \mathbf{0}$.
  while there exists a ship s with survival probability below γ_s **and** there exists an unused weapon **do**
    Sort threats $j \in T$ by $\sum_{s \in S} \alpha_s Q_{js}$ in a decreasing order.
    Pick the first (and therefore the most dangerous) threat; denote it by j_0.
    Assign unused weapon i_0 against j_0 according to $i_0 = \arg\max_{i \in H} P_{ij_0}$ and set $x_{i_0 j_0} = 1$.
    Update $Q_{j_0 s} \leftarrow Q_{j_0 s}(1 - P_{i_0 j_0})$ for $s \in S$.
  return $\mathbf{x}$, i.e., assignments of weapons to threats.

In order to demonstrate that in practice the MIO approach generates much more efficient solutions to the FDP than GREEDY method, we consider an example of the fleet defense scenario as in (Becker et al. 2013). This one time-period problem is described by the following parameters:

- Fleet: There are three ships of equal importance, i.e., $S = 3$ and $\alpha_s = 1$ for $s \in S$. Asset survival thresholds γ_s are generated uniformly between 0.7 and 0.95, and $\lambda = 1000$.
- Threats: There are nine threats in the field of action ($T = 9$). Targeting matrix $\mathbf{Q}$ is defined by nonzero elements:

$$Q_{11} = Q_{13} = Q_{41} = Q_{42} = Q_{72} = Q_{73} = 0.5 \quad \text{and} \quad Q_{21} = Q_{31} = Q_{52} = Q_{62} = Q_{83} = Q_{93} = 1.$$

- Weapons: Each ship has 5 hard-kill missiles. Components of efficacy matrix $\mathbf{P}$ are uniformly generated at random between 0.5 and 1. Firing costs r_i are equal to 1 for all weapons $i \in W$.

In this and all subsequent experiments, we coded all algorithms in Julia/JuMP (Lubin and Dunning 2015) and solved them with Gurobi 6.5 on a computer with 2.5Hz processor and 16GB of memory. We ran 1000 Monte Carlo simulations solving generated FDP instances by the GREEDY and the MIO methods and compared solutions that both methods yielded. In 46% of the scenarios the solutions turned out to be identical, while in 54% of the scenarios the solutions produced by the GREEDY method were less efficient. The GREEDY method allocated on average 14.5% more weapons for fleet defense than the MIO approach. Moreover, in 3.5% of the cases the MIO algorithm protected all the ships in the fleet with required probabilities, while the GREEDY algorithm allocated all available weapons and nevertheless some of the ships were left insecure.

5.2. Performance of the Lazy Constraints Method

This section compares the computational performance of the two formulations of the centralized fleet defense problem (1): the SOCP formulation, that can be derived from the results of paper (Becker et al. 2013), and a formulation with lazy constraints presented in Section 2.2. The lazy constraints algorithm may employ linear projection (7a) or orthogonal projection (7b) of a testing point. Therefore, in order to model the nonlinear constraints (1b) it is possible to use linear projections, orthogonal projections or both, i.e., there are three available variants. The same independent decisions can be made for constraints (1c), that gives us in total 9 different methods to solve the problem with lazy constraints techniques.

Based on the different ways to project testing points, we define the *parallelized method* as follows. For each instance of the FDP, we run 9 methods described above in parallel, and terminate computation when one the 9 arms solves the optimization problem to optimality for a predefined level of tolerance δ.

In order to demonstrate that the parallelized lazy constraint method is remarkably faster than the SOCP method we consider the same FDP scenario as described in the previous experiment. The only difference is that now we add soft-kill weapons.

- Weapons: Each ship has 5 hard-kill missiles and 1 soft-kill weapon on board. Components of the efficacy matrix $\mathbf{P}$ are generated uniformly between 0.5 and 1. A soft-kill weapon can be fired by a ship only at threats that target this ship with probability 1. This assignment destroys a threat with probability 0.3 and redirects a threat to a neighboring asset with probability 0.2. It does not make any effect with probability 0.3. Firing costs r_i are equal to 1 for all weapons $i \in W$.

We evaluated two methods to solve the FDP - the SOCP and the parallelized lazy constraints algorithm - measuring computational time needed to find an optimal weapon allocation and to prove its optimality. The results of the second-order conic formulation are presented in Table 2, while computational times for an algorithm based on lazy constraints are reported in Table 3.

The algorithm with iteratively added linear separating hyperplanes has starkly better performance than SOCP method. In this particular example, an average acceleration of finding the optimal solution of the FDP is equal to $\frac{389}{3.93} = 98.9$ times.

5.3. Sensitivity Analysis

In this experiment, we explore the influence of major parameters (survival threshold, weapons efficiency and number of threats) on the computational time needed to solve the FDP. The default setting is defined as follows:

- Fleet: There is one ship $S = 1$ with priority $\alpha_1 = 1$. Number of periods $N = 1$, threshold γ is randomly generated between 0.7 and 0.95, penalty $\lambda = 1000$.
- Threats: The total number of threats $T = 5$ and components of targeting vector $\mathbf{Q} = (Q_{j1}, \ldots, Q_{jT})$ are generated uniformly at random between 0 and 1.
- Weapons: The number of hard-kill interceptors $H = 10$ with uniformly random efficiency between 0.5 and 1. There is also one soft-kill weapon $K = 1$, which deflects the threat with probability 0.6 and does not change anything with probability 0.4.

Having fixed default parameters, we sequentially perturb each of them to find out how it affects the computational time for solving the corresponding optimization problem. For each of the scenarios we generate 1000 Monte Carlo simulations and solve them with the parallelized method. In Table 4 we report 95% quantiles of computational time needed to find the optimal solution and to prove its optimality. In other words, in 95% of all simulations the optimal solution was found and proved to be optimal faster than times indicated in the corresponding cells of the table. The time needed to obtain the first integer solution was always below 1 second, and therefore omitted.

Case 1: Changes in survival threshold γ. These empirical results allow us to make the following practical conclusion. If the survival threshold of ship s belongs, for instance, to the interval [0.9; 0.95] and the decision maker terminates the computation of the FDP after 0.6 s, then with 95 % probability the best integer solution found by this time is optimal. Furthermore, we infer that the growth of survival threshold implies the growth of computational time since it becomes more difficult to protect a ship with a higher probability.

Case 2: Changes in weapons efficiency P_{ij}. This case study validates that the growth of weapons accuracy naturally implies the simplification of the fleet defense optimization problem, what reduces the overall computational time.

Case 3: Changes in a number of threats T. The last section of Table 4 demonstrates the growth of computational time needed to solve more and more stressing scenarios with a sequentially increasing number of incoming threats.

The conducted numerical experiments make evident that the FDP can be solved by the parallelized method within a matter of seconds (that is, online) for the instances of practical size and reasonable values of underlying parameters. The more stressful the scenario becomes (described by higher survival probability thresholds, lower weapon efficiency or a larger number of incoming threats), the more computational time is needed to find the optimal solution.

5.4. Scaling Experiments

In this subsection, we discuss the scalability of the FDP formulation with respect to the size of the fleet and the total number of attacks. As opposed to the previous experiment, we now increase all major parameters of the problem proportionally to one selected parameter, rather than change a single parameter keeping the rest of them fixed. The default setting is defined as follows:

- Fleet: Survival thresholds γ_s for $s \in S$ are uniformly generated at random between 0.7 and 0.95, asset priorities α_s for $s \in S$ are random integers from 1 to 10 and $\lambda = 1000$.
- Threats: Total number of threats is defined by a vector $(T^1, \ldots, T^N)$, where N is a number of attacks. Components Q^τ_{js} of targeting matrices $\mathbf{Q}^\tau$ are randomly generated for $\tau \in \{1, \ldots, N\}$, $j \in T^\tau$ and $s \in \{1, \ldots, S\}$ in such a way, that any missile j cannot target more than 3 ships.
- Weapons: Each ship has 5 hard-kill weapons per time period with a random efficiency P^τ_{ij} between 0.6 and 1. Moreover, each ship holds one soft-kill weapon per period. Each soft-kill weapon deflects a threat from the fleet with probability 0.3, it redirects a threat to another asset with probability 0.4 and does not affect the missile with probability 0.4.

Similarly to the previous experiment, we generate 1000 Monte Carlo simulations in order to measure 95% quantile of computational times needed to find an optimal solution and to prove its optimality.

Case 1: Scaling with respect to a number of ships S. This scenario is defined by a fixed number of attacks $N = 1$ and a changing size of the fleet S. A total number of hard-kill weapons H, soft-kill weapons K as well as a total number of incoming threats T depend on S as follows:

$$H = 5 \cdot S, \quad K = S \quad \text{and} \quad T = 3 \cdot S.$$

Case 2: Scaling with respect to a number of time periods N. In this setting, we keep a number of ships S fixed and equal to 3, while the number of subsequent attacks N varies. We define major parameters of the FDP to be linear functions of N:

$$H_s = 5 \cdot N; \quad K_s = N \text{ for } s \in S; \quad T^\tau = 3 \text{ for } \tau \in \{1, \dots, N\}.$$

Table 5 demonstrates one more time that the FDP can be solved within a matter of seconds for instances of typical size with reasonable sizes of a campaign S and a number of time periods N. The results from Table 5 also imply that the computational time needed to find the optimal solution of the FDP and to prove its optimality exhibits a moderate growth as a function of S and N.

5.5. Advantage of the multiperiod approach

In this experiment, we consider the multiperiod scenario for the FDP described in Section 3. Our primary goal is to demonstrate that in this case the method based on the multiperiod formulations (9) significantly outperforms the method based on myopic single period formulations (1). More precisely, in case of a sequence of attacks (indexed by $\tau = 1, \dots, N$), the decision maker at any given time-step τ can organize the fleet defense in two different ways:

Method A. She can solve the *single period* optimization problem of type (1) that comprises data describing only the current time period τ, i.e., $\mathbf{P}^\tau, \mathbf{Q}^\tau, \mathbf{R}^\tau$.

Method B. She can solve the *multiperiod* optimization problem of type (9) that comprises data describing the current and the future time periods, i.e., $\mathbf{P}^t, \mathbf{Q}^t, \mathbf{R}^t$ for $t = \tau, \dots, N$.

The numerical experiment is defined as follows:

- Fleet: The size of the fleet $S = 4$ and ship priorities are $\alpha_s = s$ for $s = 1, \dots, S$. The number of periods $N = 3$, thresholds γ_s are randomly generated between 0.7 and 0.95, and $\lambda = 1000$.
- Threats: The number of threats per period $T^\tau = 4$ for $\tau = 1, \dots, N$. At each time-step $\tau = 1, \dots, N$, there is one threat that targets a randomly chosen ship with probability 1, while the remaining threats can target up to two ships with probabilities generated uniformly at random.

- Weapons: The number of hard-kill interceptors $H = 11$ with uniformly random efficiency between 0.5 and 1. Each ship carries one soft-kill weapon, which destroys the threat with probability 0.3, redirects it to a neighboring ship with probability 0.4 and does not change anything with probability 0.3.

At the end of each time-step $\tau = 1, \ldots, N$ we simulate a possible destruction of the ships as follows. Let us assume that ship s was protected with a probability p_s^τ. If p_s^τ is at least as high as γ_s, then we assume that this ship is secure. Otherwise, the survival of ship s at the end of time-step τ is modeled by Bernoulli random variable ξ_s^τ with parameter p_s^τ.

We compare the performance of the methods A and B in terms of four characteristics describing the quality of the FDP solution:

1. The total number of ships that survive the sequence of attacks, $\sum_{s \in S} \max\Big(\mathbb{I}\big\{w_s^N = 1\big\}, \mathbb{I}\big\{w_s^N > 1 \text{ and } \xi_s^N \le p_s^N\big\}\Big)$.
2. The total *weighted* number of surviving ships, where ship priorities are used as weights, $\sum_{s \in S} \alpha_s \cdot \max\Big(\mathbb{I}\big\{w_s^N = 1\big\}, \mathbb{I}\big\{w_s^N > 1 \text{ and } \xi_s^N \le p_s^N\big\}\Big)$.
3. The total number of weapons fired at threats, $\sum_{\tau=1}^{N} \sum_{i \in W} \sum_{j \in T^\tau} x_{ij}$.
4. The total number of weapons wasted due to the destruction of the ships.

Table 6 demonstrates the empirical results of one thousand Monte Carlo simulations. Given that Method B is exposed to more relevant information about the field of action, it is able to produce more efficient and strategic weapon assignments. We observe that the average number of surviving ships is higher and the average number of destroyed weapons is lower when Method B is employed.

5.6. Cooperation Protocol Results

In this experiment, we compare the performance of the three algorithms solving the FDP from the perspective of global fleet objectives:

1. Decentralized: In this setting ships in the fleet do not communicate with each other and execute independent self-defense plans.

2. Communication protocol: Ships in the fleet have some communication capability and cooperate according to the steps described in Section 4.

3. Centralized: All information about the field of action is globally known, so the exact solution of fleet-wide optimization problem is calculated and implemented.

Similarly to the previous experiment, we consider four characteristics that describe the quality of a FDP solution produced by an algorithm:

1. The total number of ships that survive, $\sum_{s \in S} \mathbb{I}\{w_s = 1\}$.

2. The total *weighted* number of surviving ships, where ship priorities are used as weights, $\sum_{s \in S} \alpha_s \cdot \mathbb{I}\{w_s = 1\}$.

3. The average value of a multiplicative slack variable, $\frac{1}{|S|} \sum_{s \in S} \frac{w_s - 1}{w_s}$. This value displays by how much on average a survival probability of a ship is below a corresponding survival threshold.

4. The total number of weapons spent, $\sum_{s \in S} \sum_{i \in W_s} \sum_{j \in T} x_{ij}$.

In this case study we consider a scenario defined by parameters similar to the previous experiments:

- Fleet: Number of attacks $N = 1$, the size of the campaign $S = 5$, thresholds γ_s for $s \in S$ are uniformly generated at random between 0.7 and 0.95, penalty constants $\lambda = 1000$ and $\nu = 10$, asset priorities $\alpha_s = s$ for $s = 1, \ldots, S$. Each ship can send up to 2 messages at Step 1 of the communication protocol, i.e., $\hat{F}_s = 2$ for $s = 1, \ldots, S$.

- Threats: Total number of threats T equals 5, and elements of a targeting matrix $\mathbf{Q}$ are randomly generated in such a way that every missile may target up to 5 ships.

- Weapons: Each asset has only hard-kill interceptors with random effectiveness parameters between 0.5 and 1. This makes minimum weapon accuracy δ_s equal to 0.5 for all ships $s \in S$. A total number of weapons H equals 15, and these weapons are randomly split between the ships of the fleet. Firing costs r_i are set to 1 for all interceptors.

One thousand of Monte Carlo runs reveal the comparative performance presented in Table 7.

A completely distributed approach with ships solving their local survival problems without any communication with each other has the worst performance in terms of fleet-wide objectives. In this setting, on average only 2.15 out of 5 ships can protect themselves with the required probabilities. The weighted sum of secure ships is 6.56 out of perfect $15 = 1 + \cdots + 5$, and each ship has a survival probability which is on average 28.1% below the required threshold. According to the decentralized algorithm, the fleet spends on average 12.6 weapons to organize its defense.

At the same time, the implementation of the cooperation protocol with even very limited communication among the ships significantly improves the global objectives of the fleet. More precisely, a number of ships that can be classified as secure grows by more than one ship $(3.2 - 2.15 = 1.05)$, and this additional ship has one of the highest priorities in the fleet $(9.92 - 6.56 = 3.36)$. An average survivability slack also improves substantially from 28.1% to 10.3%. On the other hand, a total number of weapons needed to protect the fleet goes up from 12.6 to almost maximum possible 14.96 weapons, since now ships try to assist insecure companions after protecting themselves.

Table 7 demonstrates that the best algorithm for solving the FDP is the idealistic centralized approach. In order to protect the fleet from 5 threats it takes on average 11.58 weapons. Moreover, almost all 5 ships of the fleet survive with a negligibly small average value of the multiplicative slack. At the same time, the centralized method has two major issues that make it impractical. First, it does not take into consideration the local availability of some data. Second, it is relatively slow from a tractability perspective. The communication protocol suggested in Section 4 and the lazy constraint algorithm introduced in Section 2 considerably alleviate both these limitations.

5.7. Discussion

The major contributions of this numerical section can be summarized as follows:

1. By definition, the MIO approach to the FDP allocates available weapons in the most efficient way. Hence, this approach not only saves some of the weapons in comparison with another reasonable heuristic algorithm, but also significantly increases the survival probabilities of the ships.

2. Despite the fact that the initial optimization problem for the FDP is highly nonlinear, the lazy constraints algorithm is computationally effective. It is almost two orders of magnitude faster than the SOCP approach that was employed before.

3. Implementation of the lazy constraint method allows us to solve the FDP in real time for instances of typical size and reasonable values of major parameters. We can solve instances with several time periods and large campaign size within seconds as well. Moreover, in case of several attacks, utilization of the multiperiod formulation generates more efficient weapon assignments.

4. The suggested optimization based cooperation protocol provides a significant improvement with respect to a decentralized setting even with small communication capacities. It substantially increases the survival probabilities of the ships and a number of surviving ships.

6. Conclusion

We considered the FDP whose objective is to assign available hard-kill and soft-kill weapons in order to protect ships from incoming threats. The exact formulation has a form of a highly nonlinear mixed-binary optimization problem. This problem admits a SOCP reformulation that can be solved within minutes for instances of practical size. In order to expedite the process and be able to solve the FDP online, we developed a new approach employing lazy constraints techniques.

We also designed a new extended MIO formulation for the multiperiod fleet defense problem, that is characterized by a sequence of independent attacks. In this scenario, the multiperiod optimization problem takes into account more information about the field of action and therefore improves the fleet-wide objectives of protecting the ships while minimizing the weapon costs.

We considered not only the centralized version of the FDP, but also its decentralized counterpart. For this setting we developed a cooperation protocol allowing the ships some communication and leveraging robust optimization. For this protocol, we were able to design two auxiliary optimization problems whose solutions uniquely determine information that ships should broadcast and how they should exploit additional data from companions to update their local plans.

Finally, we conducted extensive numerical experiments and empirically demonstrated that both the single and multiperiod formulations of the FDP can be solved online for instances of typical size with reasonable values of underlying parameters. We also showed effectiveness of the suggested RO-based cooperation protocol in comparison with a completely distributed setting with no communication among ships and the idealistic centralized coordination.

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References

Ahuja RK, Kumar A, Jha KC, Orlin JB (2007) Exact and heuristic algorithms for the weapon-target assignment problem. *Operations Research* 55(6):1136–1146.

Becker A, Bertsimas D, Cofield J, Finkel D, Son SH, Stigile J (2013) Weapon coordination in a fleet: Centralized and distributed optimization, MIT Lincoln Laboratory. Working paper.

Ben-Tal A, El Ghaoui L, Nemirovski A (2009) *Robust optimization* (Princeton University Press).

Bertsekas DP (1990) The auction algorithm for assignment and other network flow problems: A tutorial. *Interfaces* 20(4):133–149.

Bertsimas D, Brown DB, Caramanis C (2011) Theory and applications of robust optimization. *SIAM review* 53(3):464–501.

Bisht S (2004) Hybrid genetic-simulated annealing algorithm for optimal weapon allocation in multilayer defence scenario. *Defence Science Journal* 54(3):395.

Bogdanowicz Z, Coleman N (2007) Optimization of sensor/weapon-target pairings based on auction algorithm. *WSEAS Transactions on Mathematics* 6(6):730.

Bogdanowicz Z, Coleman N (2008a) Advanced algorithm for optimal sensor-target and weapon-target pairings in dynamic collaborative engagement. Technical report, DTIC Document.

Bogdanowicz ZR, Coleman NP (2008b) New algorithm for optimization of effect-based weapon-target pairings. *CSC*, volume 8, 3–8.

Boyd S, Vandenberghe L (2004) *Convex optimization* (Cambridge university press).

Cai H, Liu J, Chen Y, Wang H (2006) Survey of the research on dynamic weapon-target assignment problem. *Journal of Systems Engineering and Electronics* 17(3):559–565.

Cheung Y, Chung JH (2010) Semi-autonomous collaborative control for multi-weapon multi-target pairing. *Control Automation and Systems (ICCAS), 2010 International Conference on*, 540–545 (IEEE).

Gurobi Optimization Inc (2016) Gurobi optimizer reference manual. URL http://www.gurobi.com.

Hosein P, Athans M (1990) Preferential defense strategies. The static case. Technical report, MIT LIDS.

IBM Knowledge Center (2017) Differences between user cuts and lazy constraints. http://www.ibm.com/support/knowledgecenter/SSSA5P_12.5.1/ilog.odms.cplex.help/CPLEX/UsrMan/topics/progr_adv/usr_cut_lazy_constr/04_diffs.html, accessed: 2017-04-01.

Leboucher C, Shin H, Siarry P, Chelouah R, Menec S, Tsourdos A (2013) A two-step optimisation method for dynamic WTA problem. *InTech.* 10.

Lobel I, Ozdaglar A, Feijer D (2011) Distributed multi-agent optimization with state-dependent communication. *Mathematical Programming* 129(2):255–284.

Lubin M, Dunning I (2015) Computing in OR using Julia. *INFORMS Journal on Computing* 27(2):238–248.

Malhotra A, Jain R (2001) Genetic algorithm for optimal weapon allocation in multilayer defence scenario. *Defence science journal* 51(3):285.

Manne AS (1958) A target-assignment problem. *Operations Research* 6(3):346–351.

Maza I, Caballero F, Capitan J, Martinez-de Dios JR, Ollero A (2011) A distributed architecture for a robotic platform with aerial sensor transportation and self-deployment capabilities. *Journal of Field Robotics* 28(3):303–328.

Murphey RA (2000) Target-based WTA problems. *Nonlinear Assignment Problems*, 39–53 (Springer).

Ren W, Sorensen N (2008) Distributed coordination architecture for multi-robot formation control. *Robotics and Autonomous Systems* 56(4):324–333.

Sariel S, Balch T (2006) A distributed multi-robot cooperation framework for real time task achievement. *Distributed Autonomous Robotic Systems 7*, 187–196 (Springer).

Terelius H, Topcu U, Murray RM (2011) Decentralized multi-agent optimization via dual decomposition. *IFAC Proceedings Volumes* 44(1):11245–11251.

Table 1: Notation.

A. Indices	
s, s', s''	Ship index
i	Weapon index
j, k	Threat index
τ	Time-step, attack number
B. Sets	
S	Ships in a fleet
T^{τ}	Incoming threats at time-step τ
T_s	Incoming threats that target ship s with non-zero probability
W	All available weapons
H	Hard-kill weapons
H_s	Hard-kill weapons on board ship s
K	Soft-kill weapons
K_s	Soft-kill weapons on board ship s
A_{ij}^{τ}	Set of threats affected if at time τ threat j is engaged by soft-kill weapon i
B_k^{τ}	Set of all soft-kill interactions at time-step τ that affect threat k
$U_{js'}(s)$	Uncertainty set for targeting probability $\tilde{Q}_{js'}(s)$
$\mathbf{U}(s)$	Cartesian product of sets $U_{js'}(s)$ with respect to j and s'
$\mathbf{f}^*$	Optimal values of communication decision variables
C. Data and Parameters	
Q_{js}^{τ}	Probability that at time-step τ threat j targets ship s
P_{ij}^{τ}	Probability that at time-step τ hard-kill weapon i destroys threat j, if i is fired against j

$R^{\tau}(j,s\,\|\,i,k)$	Probability that at time-step τ threat j targets ship s, if soft-kill weapon i is fired against threat k
γ_s	Asset survivability threshold
α_s	Asset priority value
r_i	Firing cost of weapon i
λ	Sufficiently large penalty for lack of survival probability
N	Number of time-steps (attacks)
M	Large enough number
δ_s	Minimum accuracy of weapons of ship s
$\tilde{Q}_{js'}(s)$	Uncertain probability that threat j targets ship s' after the first stage of the protocol computed from the perspective of ship s
$\hat{Q}_{js'}(s)$	Upper bound on probability $\tilde{Q}_{js'}(s)$
$\hat{F}_s$	Maximum number of messages that ship s can broadcast
D. Decision variables	
x_{ij}^{τ}	Binary decision to fire weapon i against threat j at time-step τ
y_j^{τ}	Binary decision to have no soft-kill interactions with threat j at time-step τ
D_{js}^{τ}	Probability that at time-step τ threat j will destroy ship s
w_s^{τ}	Multiplicative slack variable for survival constraint of ship s at time-step τ
u_s^{τ}	Binary destruction indicator equals 1 if ship s is destroyed before time-step τ
f_{js}	Binary decision of ship s to broadcast a message that it fires at least one weapon

Table 2 Computational time of the SOCP method.

	Average time	Minimum time	Maximum time
First integer solution	32 s	3 s	73 s
Optimal solution found	**389** s	3 s	570 s
Optimality is proven	545 s	125 s	956 s

Table 3 Computational time of the parallelized lazy constraints method.

	Average time	Minimum time	Maximum time
First integer solution	0.19 s	0.01 s	2.1 s
Optimal solution found	**3.93** s	0.4 s	7.4 s
Optimality is proven	16.91 s	4.1 s	27.8 s

Table 4 Sensitivity of the computational time with respect to problem parameters

Case 1: Range for γ	$\gamma \in [0.7; 0.75]$	$\gamma \in [0.7; 0.95]$	$\gamma \in [0.9; 0.95]$	$\gamma \in [0.95; 1]$
95% Quantile (Opt found)	0.17 s	0.43 s	**0.6** s	10.7 s
95% Quantile (Opt proven)	0.37 s	1.53 s	3.7 s	15.7 s
Case 2: Range for P_{ij}	$P_{ij} \in [0.5; 0.7]$	$P_{ij} \in [0.6; 1]$	$P_{ij} \in [0.75; 1]$	$P_{ij} \in [0.9; 1]$
95% Quant. (Opt found)	9.41 s	0.39 s	0.46 s	0.12 s
95% Quant. (Opt proven)	30.1 s	2.39 s	1.67 s	0.32 s
Case 3: Number of threats T	$T = 3$	$T = 5$	$T = 7$	$T = 9$
95% Quantile (Opt found)	0.06 s	3.93 s	14.7 s	26.4 s
95% Quantile (Opt proved)	0.11 s	16.9 s	28.1 s	30.2 s

Table 5 Computational time of the parallelized method.

Size of the fleet S	$S = 2$	$S = 4$	$S = 6$	$S = 8$
95% Quantile (Opt found)	1.23 s	6.01 s	15.4 s	22.8 s
95% Quantile (Opt proved)	25.4 s	30.3 s	41.3 s	63 s

Number of attacks N	$N = 1$	$N = 2$	$N = 3$
95% Quantile (Opt found)	0.15 s	0.84 s	1.81 s
95% Quantile (Opt proved)	0.26 s	2.42 s	8.54 s

Table 6 Comparison of methods solving the multiperiod FDP

Method	Ships survived	Weighted ships	Weapons fired	Weapon wasted
A: Single period	3.17	8.1	13.15	0.45
B: Multiperiod	3.59	8.95	13.37	0.17

Table 7 Comparison of algorithms solving the FDP

	Ships survived	Weighted ships	Average slack	Total weapons
Decentralized	2.15	6.56	0.281	12.6
Protocol	3.2	9.92	0.103	14.96
Centralized	4.94	14.9	7e-4	11.58

Proofs of Statements

EC.1. Proof of Lemma 1.

Proof of Lemma 1. **Case 1** (7a). Let us introduce the function $f(\mathbf{x}) = \prod_{i=1}^{n} x_i - \gamma$, for $\mathbf{x} \in \mathbb{R}_+^n$. If point $\mathbf{y} \in \mathbb{R}_+^n$ does not belong to set Σ, then it is below the graph of the function $f(\mathbf{x})$ (Fig. 1a). Hence, we consider point $\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$ such that the components of $\hat{\mathbf{y}}$ are proportional to the components of $\mathbf{y}$, and point $\hat{\mathbf{y}}$ belongs to the surface $f(\mathbf{x}) = 0$, that is, the boundary of convex set Σ. In other words, for some $t > 0$ the following system of equations holds:

$$\begin{cases} \hat{y}_i = y_i t, \quad i = 1, \ldots, n \\ \prod_{i=1}^{n} \hat{y}_i - \gamma = 0. \end{cases}$$

We denote the unique solution of this system of equations with respect to variable t as

$$t^* = \sqrt[n]{\frac{\gamma}{y_1 y_2 \ldots y_n}} > 1.$$

We next find a tangent hyperplane to the smooth function $f(\mathbf{x})$ at point $\hat{\mathbf{y}}$ according to the formula

$$\nabla f(\hat{\mathbf{y}})^\top (\mathbf{x} - \hat{\mathbf{y}}) = 0.$$

Since $\frac{\partial f}{\partial x_i}(\hat{\mathbf{y}}) = \prod_{j \neq i} \hat{y}_j = \frac{\gamma}{\hat{y}_i}$ for all $i = 1, \ldots, n$, then the hyperplane has the form

$$\sum_{i=1}^{n} \frac{\gamma}{\hat{y}_i}(x_i - \hat{y}_i) = 0,$$

what is equivalent to (6). This tangent hyperplane indeed separates point $\mathbf{y}$ from the convex set Σ considering

$$\sum_{i=1}^{n} \frac{\gamma}{\hat{y}_i}(y_i - \hat{y}_i) = \sum_{i=1}^{n} \frac{\gamma \cdot y_i}{\hat{y}_i}(1 - t^*) = \sum_{i=1}^{n} \frac{\gamma\,(1 - t^*)}{t^*} < 0.$$

Case 2 (7b). The orthogonal projection $\hat{\mathbf{y}}$ of the point $\mathbf{y}$ onto the convex set Σ is the unique solution of the optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|_2^2 \\ \text{s.t.} \quad & x_1 x_2 \ldots x_n = \gamma \\ & x_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

The Lagrangian function for this problem has the form

$$L(\mathbf{x},\mu)=\frac{1}{2}\sum_{i=1}^{n}(x_i-y_i)^2+\mu(x_1x_2\dots x_n-\gamma).$$

The Karush-Kuhn-Tucker optimality conditions (Boyd and Vandenberghe 2004) imply that

$$x_i=y_i-\mu^*\frac{\gamma}{x_i},\quad x_i>0,\quad \text{for } i=1,\dots,n,$$

while a non-positive optimal Lagrange multiplier μ^* can be found from the equation

$$\prod_{i=1}^{n}x_i=\prod_{i=1}^{n}\frac{y_i+\sqrt{y_i^2-4\mu\gamma}}{2}=\gamma.$$

Since $y_i\geq 0$ for $i=1,\dots,n$ and $\gamma>0$, it is easy to see that for $\mu\leq 0$ the function $g(\mu)$ defined in (8) is strictly monotonically decreasing as a composition of decreasing functions. Moreover, $g(0)=\prod_{i=1}^{n}y_i-\gamma<0$ and

$$g\big(-\gamma^{\frac{2}{n}-1}\big)=\prod_{i=1}^{n}\frac{y_i+\sqrt{y_i^2+4\gamma^{\frac{2}{n}}}}{2}-\gamma\geq\prod_{i=1}^{n}\frac{\sqrt{4\gamma^{\frac{2}{n}}}}{2}-\gamma\geq 0.$$

Thus, there is a unique solution μ^* of the equation $g(\mu)=0$ in the interval $[-\gamma^{\frac{2}{n}-1};0]$, that can be found numerically with binary search techniques. Having found the optimal value of Lagrange multiplier μ^*, we determine the orthogonal projection $\hat{\mathbf{y}}$ in closed form (7b). Finally, we construct a tangent hyperplane to the convex set Σ at $\hat{\mathbf{y}}$ of the form (6). Separation of point $\mathbf{y}$ by the tangent hyperplane from Σ is due to the inequality

$$\sum_{i=1}^{n}\frac{\gamma}{\hat{y}_i}(y_i-\hat{y}_i)=\sum_{i=1}^{n}\frac{2\gamma}{y_i+\sqrt{y_i^2-4\mu^*\gamma}}\frac{y_i-\sqrt{y_i^2-4\mu^*\gamma}}{2}=\sum_{i=1}^{n}\frac{\Big(y_i-\sqrt{y_i^2-4\mu^*\gamma}\Big)^2}{4\mu^*}<0,$$

that completes the proof since μ^* is strictly negative. □

EC.2. Proof of Proposition 1.

Proof of Proposition 1. In order to prove (17) we will show that the optimal solution $(\mathbf{x}^*,\mathbf{w}^*,\mathbf{f}^*,\mathbf{D}^*)$ of optimization problem (11) is feasible for problem (14). Combining definitions (12) and (13) we obtain that

$$\tilde{Q}_{js'}(s)\leq\hat{Q}_{js'}(s)\leq Q_{js},\quad j\in T,\;\; s',s\in S. \tag{EC.1}$$

For any fixed index $j \in T_s$, Ineq. (11b) defining destruction probability D_{js} at optimum has the following form

$$D^*_{js} \geq Q_{js}\Big(\prod_{i\in H_s}\Big(1+\frac{P_{ij}\,x^*_{ij}}{1-P_{ij}}\Big)\Big)^{-1}.$$

If targeting probabilities Q_{js} are substituted with lower values $\tilde{Q}_{js}(s)$ as in (14b), then the current solution $\mathbf{D}^*$ is still feasible due to

$$D^*_{js} \geq Q_{js}\Big(\prod_{i\in H_s}(1+\frac{P_{ij}\,x^*_{ij}}{1-P_{ij}})\Big)^{-1} \geq \tilde{Q}_{js}(s)\Big(\prod_{i\in H_s}(1+\frac{P_{ij}\,x^*_{ij}}{1-P_{ij}})\Big)^{-1},$$

which holds for all $\tilde{Q}_{js}(s) \in U_{js}(s)$ defined by (12).

Now we need to prove that the destruction probabilities $D_{js'}$ defined by Eq. (11c) are also feasible for inequalities (14b). Indeed, if $f^*_{js} = 0$, then inequality

$$1 \leq \Big(\tilde{Q}_{js'}(s)\Big)^{-1} Q_{js'}(1-\delta_s f^*_{js})\prod_{i\in H_{s'}}\Big(1+\frac{P_{ij}\,x^*_{ij}}{1-P_{ij}}\Big) \tag{EC.2}$$

is straightforward, given that (EC.1) is true and each factor in the last product is greater or equal than 1. If $f^*_{js} = 1$, then Ineq. (11e) states that there exist at least one weapon, say $i_0 \in H_s$, that ship s fires at threat j, i.e. $x^*_{i_0 j} = 1$. In this case, Ineq. (EC.2) still holds, since

$$(1-\delta_s f^*_{js})\prod_{i\in H_{s'}}\Big(1+\frac{P_{ij}\,x^*_{ij}}{1-P_{ij}}\Big) \geq (1-\delta_s)\Big(1+\frac{P_{i_0 j}}{1-P_{i_0 j}}\Big) = \frac{1-\delta_s}{1-P_{i_0 j}} \geq 1,$$

by definition of minimum accuracy of hard-kill weapons δ_s.

The rest of the constraints in (14) are identical to formulation (11) and, therefore, remain valid. □