J. Appl. Prob. **28**, 873–885 (1991)
Printed in Israel

TRANSIENT AND BUSY PERIOD ANALYSIS OF THE $GI/G/1$ QUEUE AS A HILBERT FACTORIZATION PROBLEM

DIMITRIS J. BERTSIMAS,* *Sloan School of Management, MIT*
JULIAN KEILSON,* *Sloan School of Management, MIT*
DAISUKE NAKAZATO,** *Operations Research Center, MIT*
HONGTAO ZHANG,** *Operations Research Center, MIT*

Abstract

In this paper we find the waiting time distribution in the transient domain and the busy period distribution of the $GI/G/1$ queue. We formulate the problem as a two-dimensional Lindley process and then transform it to a Hilbert factorization problem. We achieve the solution of the factorization problem for the $GI/R/1$, $R/G/1$ queues, where R is the class of distributions with rational Laplace transforms. We obtain simple closed-form expressions for the Laplace transforms of the waiting time distribution and the busy period distribution. Furthermore, we find closed-form formulae for the first two moments of the distributions involved.

TRANSIENT ANALYSIS; BUSY PERIOD; LINDLEY EQUATION

1. Introduction

Transient and busy period analyses in queueing models have long been considered as very difficult problems, although in many situations it is very important to study the transient behavior of queueing systems. For example, systems often encounter transient behavior due to exogenous changes, such as the opening or closing of a queueing system or the application of a new control. Furthermore, even in systems with time-homogeneous behavior the convergence to steady state is so slow that the equilibrium behavior is not indicative of system behavior. Examples from practical situations in which transient phenomena are important include manufacturing systems with frequent start-up periods and transportation systems with time-varying demand (for example, airport runway operations in major airports).

In this paper, we derive simple closed-form expressions for the Laplace transforms of the waiting time distribution under FCFS when the system is initially empty and the busy period distribution for the $GI/G/1$ queue. We first formulate the problem as a

Received 12 December 1989; revision received 5 March 1991.
Postal address: *Sloan School of Management and **Operations Research Center, MIT, Cambridge, MA 02139, USA.
The research of Dr Bertsimas was partially supported by grants from the Leaders for Manufacturing program at MIT and from Draper Laboratory.

two-dimensional Lindley process and then transform it to a Hilbert factorization problem. We are able to solve explicitly the underlying factorization problem. Due to the numerical tractabilty of the solution, we put special emphasis on the cases of $GI/R/1$ and $R/G/1$ queues, where R is the class of distributions with rational Laplace transforms.

Formulations of queueing problems as Hilbert factorization problems can be traced back in Lindley [10], in which the steady-state waiting-time distribution of the $GI/G/1$ queue is derived via a spectral factorization of the underlying Hilbert problem. For other examples of the method see Keilson [6], [7].

In recent years, work on the transient behavior of queueing systems has concentrated on numerical techniques. This change in emphasis was primarily motivated by the analytical complexity of the problems involved. The two principal methods are the randomization technique introduced by Jensen [5] (see Grassmann [3]) and numerical integration methods of the underlying Kolmogorov differential equations (see Gross and Harris [4], Section 7.3.2 and references therein). Analytical investigations of the transient behavior of queueing systems are in general rare. For the $M/M/1$ queue, expressions for the queue length probabilities are known as sums of modified Bessel functions (see Gross and Harris [4]). Ramaswami [11] has characterized the busy period of a $G/PH/1$ queue using the matrix geometric approach.

The paper is organized as follows. In the next section, which is central to the paper, we formulate the transient behavior of the $GI/G/1$ queue as a two-dimensional Lindley process, derive the key formula of the transient and busy period dynamics and then transform it to a Hilbert factorization problem. In Section 3, we solve the factorization problem for the $R/G/1$ queue, while in Section 4 we achieve its solution for the $GI/R/1$ queue. In Section 5 we observe how the results of the previous two sections are in agreement with the known results for the $M/G/1$ and $GI/M/1$ queues and consistent with the results of Bertsimas and Nakazato [2]. The final section contains some closing remarks.

2. System formulation

In this section we formulate the transient behavior of the $GI/G/1$ queue as a two-dimensional Lindley process, derive the key formula of the transient dynamics and then transform it to a Hilbert factorization problem. Our analysis will focus on the notion of a *busy cycle*, which is defined as the busy period plus an immediately following idle period. In Subsection 2.1 we define our notation, in Subsection 2.2 we derive the key formula for the transient dynamics and in Subsection 2.3 we transform the problem to a Hilbert factorization problem.

2.1. *Notation and assumptions*. In this subsection we define the random variables and establish the notation we are using. We assume that the system is initially idle and the first customer's arriving time is the forward recurrence time of the arrival process. Although this assumption is restrictive for the waiting time distribution, it is not restrictive for the busy period distribution, since the busy period regenerates.

We first define the following random variables:
X_n: the service time of the nth customer.
T_n: the interarrival time between the $(n-1)$th and nth customer.
τ_n: the arriving time of the nth customer. Note that $\tau_n = \tau_1 + \Sigma_{k=2}^{n} T_k$.
τ: the arriving time of a random customer.
B_I: the duration of a busy cycle, i.e. the interval between the initiating epoch of a busy period and the initiating epoch of the next busy period.
B_P: the duration of a busy period.
W_n^+: the waiting time in the queue of the nth customer.
W^+: the waiting time of a random customer.

We use the following notation:
$a(t)$: the interarrival time probability density function (p.d.f.).
$\alpha(s)$: the Laplace transform of $a(t)$.
$1/\lambda = E[T_n] = -\dot{\alpha}(0)$: the mean interarrival time.
$C_T^2 = \mathrm{Var}[T_n]/E[T_n]^2$: the squared coefficient of variation of the interarrival time.
$a^*(t)$: the first customer's arriving time p.d.f. (because of our assumption it is the forward recurrence time of the interarrival time).
$\alpha^*(s)$: the Laplace transform of $a^*(t)$, i.e. $\alpha^*(s) = (\lambda/s)(1-\alpha(s))$.
$b(t)$: the service time p.d.f.
$\beta(s)$: the Laplace transform of $b(t)$.
$1/\mu = E[X_n] = -\dot{\beta}(0)$: the mean service time.
$C_X^2 = \mathrm{Var}[X_n]/E[X_n]^2$: the squared coefficient of variation of the service time.
$\rho = \lambda/\mu$: the traffic intensity.
$s_I(t)$: the busy cycle p.d.f.
$s_P(t)$: the busy period p.d.f.
$\sigma(s)$: the Laplace transform of $s_P(t)$.

In addition, we define

$$f(x,y) = \frac{\partial}{\partial y}\Pr[W^+ \leqq y \mid \tau = x]$$

$$= \lim_{N\to\infty} \frac{\dfrac{1}{N}\sum_{n=1}^{N} \dfrac{\partial^2}{\partial x \partial y}\Pr[\tau_n \leqq x,\, W_n^+ \leqq y]}{\dfrac{1}{N}\sum_{n=1}^{N}\dfrac{d}{dx}\Pr[\tau_n \leqq x]} \tag{1}$$

$$= \frac{\dfrac{\partial^2}{\partial x \partial y}\sum_{n=1}^{\infty}\Pr[\tau_n \leqq x,\, W_n^+ \leqq y]}{\dfrac{d}{dx}\sum_{n=1}^{\infty}\Pr[\tau_n \leqq x]}.$$

2.2. *Transient dynamics.* In this subsection we derive the key formula that describes the transient dynamics of the $GI/G/1$ queue. For notational convenience we enumerate customers by 0, 1, 2,$\cdots$, n in the order of arrival. We analyze the case in which the nth customer arrives at the busy period initiated by the kth customer. We let

$$\xi_n = X_{n-1} - T_n, \qquad W_n = \sum_{r=k+1}^{n} \xi_r$$

and observe (see Figure 1) that if $W_{n+k} \leqq 0$ and $W_r > 0$ for $r = k + 1 \cdots n + k - 1$ then

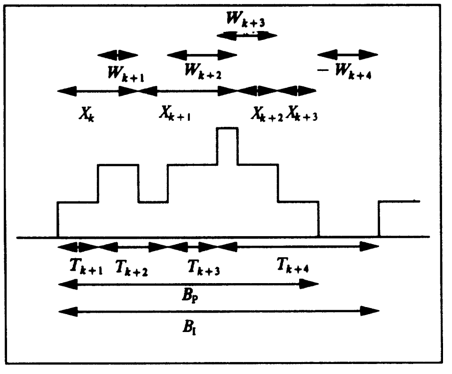

Figure 1. Transient dynamics

$$B_I = \sum_{r=k+1}^{k+n} T_r = \tau_{n+k} - \tau_k,$$

$$\text{(2)} \qquad B_P = \sum_{r=k}^{k+n-1} X_r = \sum_{r=k+1}^{k+n} (T_r + \xi_r) = B_I + W_{n+k}.$$

Similarly,

(3) if $W_r > 0$ for $r = k + 1, \cdots, n + k$, then $W_{n+k}^{+} = W_{n+k}$.

Summarizing, the critical observation is that if $W_{n+k} \leqq 0$, then the idle period immediately following the busy period B_P is $-W_{n+k}$; on the other hand, if $W_{n+k} > 0$, then W_{n+k} is the waiting time of the $(k + n)$th customer. Therefore, if we keep track of the busy cycle B_I and the quantity W_{n+k}, then we can find both the busy period and the waiting time from (2) and (3) respectively. For this goal we now consider the joint densities:

$$\Delta(x, y) = \frac{\partial^2}{\partial x \partial y} \Pr\{T_n \leqq x, \xi_n \leqq y\},$$

$$f_n(x, y) = \frac{\partial^2}{\partial x \partial y} \Pr\{\tau_{n+k} - \tau_k \leqq x, W_{n+k}^{+} \leqq y, W_r > 0, r = k + 1, \cdots, n + k\},$$

$$f_0(x, y) = \delta(x)\delta(y),$$

where $\delta(x)$ is the Dirac delta function. Note that the $\Delta(x, y)$ is independent of n and $f_n(x, y)$ has positive support in y and non-negative support in x, and is independent of k.

Since $\tau_{n+k+1} - \tau_k = \tau_{n+k} - \tau_k + T_{n+k+1}$ and $W^+_{n+k+1} = W^+_{n+k} + \xi_{n+k+1}$ if $W_r > 0$, $r = k+1, \cdots, n+k+1$ we obtain the recurrence relations:

$$\begin{aligned}
&f_0(x, y) = \delta(x)\delta(y) \\
&f_1(x, y) = \Delta(x, y)U(y) \\
(4) \qquad &\quad \vdots \\
&f_{n+1}(x, y) = [f_n(x, y) * \Delta(x, y)]U(y),
\end{aligned}$$

where $U(y)$ is a unit step function and the asterisk denotes the two-dimensional convolution sign, that is $f_n(x, y) * \Delta(x, y) \triangleq \int_{-\infty}^{y} \int_0^x f_n(x-u, y-v)\Delta(u, v)du\, dv$. We also define

$$r_n(x, y) = \frac{\partial^2}{\partial x \partial y} \Pr\{B_\mathrm{I} \leqq x,\ W_{n+k} \leqq y,\ W_r > 0, r = k+1, \cdots, n+k-1,\ W_{n+k} \leqq 0\}.$$

Note that $r_n(x, y)$ has non-positive support in y and non-negative support in x and it is independent of k.

The motivation for the above definitions is that we can express the p.d.f. of the quantities of interest in terms of the functions $r_n(x, y)$. Clearly

$$s_\mathrm{I}(x) \triangleq \frac{d}{dx} \Pr\{B_\mathrm{I} \leqq x\} = \int_{-\infty}^{0} \sum_{n=1}^{\infty} r_n(x, y)dy, \tag{5}$$

and using (2),

$$s_\mathrm{P}(x) \triangleq \frac{d}{dx} \Pr\{B_\mathrm{P} \leqq x\} = \int_{-\infty}^{0} \sum_{n=1}^{\infty} r_n(x-y, y)dy. \tag{6}$$

Using (2) and (3) we obtain in a similar way as before:

$$\begin{aligned}
&r_1(x, y) = \Delta(x, y)(1 - U(y)) \\
(7) \qquad &\quad \vdots \\
&r_{n+1}(x, y) = [f_n(x, y) * \Delta(x, y)](1 - U(y)).
\end{aligned}$$

From (4) and (7) we obtain the key formula for the $GI/G/1$ transient dynamics in real time:

$$f_{n+1}(x, y) + r_{n+1}(x, y) = f_n(x, y) * \Delta(x, y). \tag{8}$$

2.3. *Formulation as a Hilbert problem*. In this subsection we work in the transform domain, where the solution of (8) is equivalent to a Hilbert factorization problem. We introduce the Laplace transforms

$$\Phi^+(s, \omega) = \int_0^\infty \int_0^\infty \exp(-sx - \omega y) \sum_{n=0}^{\infty} f_n(x, y)dx\, dy,$$

$$\rho^-(s,\omega)=\int_{-\infty}^{0}\int_{0}^{\infty}\exp(-sx-\omega y)\sum_{n=1}^{\infty} r_n(x,y)dx\,dy.$$

Note that

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\exp(-sx-\omega y)\Delta(x,y)dx\,dy=\alpha(s-\omega)\beta(\omega).$$

The superscript $+$ is employed to indicate that $\Phi^+(s,\omega)$ is analytic in the right half of the complex ω plane. Similarly, the superscript $-$ indicates that $\rho^-(s,\omega)$ is analytic in the left half of the complex ω plane.

By taking transforms in (8) we obtain

$$\Phi^+(s,\omega)+\rho^-(s,\omega)=1+\alpha(s-\omega)\beta(\omega)\Phi^+(s,\omega),$$

or equivalently

$$\Phi^+(s,\omega)(1-\alpha(s-\omega)\beta(\omega))=1-\rho^-(s,\omega). \tag{9}$$

Equation (9) is a Hilbert factorization problem in ω with fixed s, where

$\Phi^+(s,\omega)$ is analytic in $\mathrm{Re}(\omega)>0$ and $\mathrm{Re}(s)>0$,

$\rho^-(s,\omega)$ is analytic in $\mathrm{Re}(\omega)<0$ and $\mathrm{Re}(s)>0$.

The following additional boundary conditions complete the description of the factorization problem:

$$\Phi^+(s,\infty)=1\left(\Leftrightarrow\int_{0^-}^{0^+}\sum_{n=0}^{\infty} f_n(x,y)dy=\delta(x)\right)$$

$$\dot\alpha(0)<\dot\beta(0)\qquad(\Leftrightarrow\rho<1).$$

Once $\rho^-(s,\omega)$ is found, we can use (6) to obtain the Laplace transform of the busy period:

$$\sigma(s)\triangleq\int_{0}^{\infty}e^{-sx}s_{\mathrm{P}}(x)dx=\rho^-(s,s), \tag{10}$$

and similarly from (5)

$$\int_{0}^{\infty}e^{-sx}s_{\mathrm{I}}(x)dx=\rho^-(s,0).$$

The transform of the conditional waiting time (transform variable ω) in the queue of a customer whose arriving time (transform variable s) is given, can be found from $\Phi^+(s,\omega)$ as follows. From (1) we find that (where the convolution is with respect to x)

$$f(x,y)=\frac{\partial}{\partial y}\Pr[W^+\leqq y\mid\tau=x]$$

(11)

$$=\frac{1}{\lambda}a^*(x)*\sum_{r=0}^{\infty}s_1^{(r)}(x)*\sum_{n=0}^{\infty}f_n(x,y),$$

since we assumed that the arriving time of the first customer is the forward recurrence interarrival time and thus from the renewal theorem (or by simply taking Laplace transforms) we have

$$\frac{d}{dx}\sum_{n=1}^{\infty}\Pr[\tau_n\leqq x]=a^*(x)*\sum_{n=0}^{\infty}a^{(n)}(x)=\lambda,$$

and moreover

$$\frac{\partial^2}{\partial x\partial y}\sum_{n=1}^{\infty}\Pr[\tau_n\leqq x,\ W_n^+\leqq y]=a^*(x)*\sum_{r=0}^{\infty}s_1^{(r)}(x)*\sum_{n=0}^{\infty}f_n(x,y).$$

By defining

$$\Phi(s,\omega)=\int_0^{\infty}\int_0^{\infty}\exp(-sx-\omega y)f(x,y)dx\,dy$$

and taking transforms in (11) we obtain that

$$\Phi(s,\omega)=\frac{\alpha^*(s)}{\lambda(1-\rho^-(s,0))}\Phi^+(s,\omega)$$

(12)

$$=\frac{\Phi^+(s,\omega)}{s\Phi^+(s,0)}.$$

Therefore, we can express both the transforms of the busy period and the waiting time distribution in terms of $\Phi^+(s,\omega)$ and $\rho^-(s,\omega)$. As a result, we have reduced the problem of obtaining the transforms of the busy period and the waiting time distribution to the solution of the Hilbert problem (9).

A general solution for the Hilbert problem (9) can be expressed in terms of Cauchy–Fourier integration as follows:

$$\sigma(s)=1-\exp\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C^+}\log(x)\frac{\partial}{\partial x}\log(1-\alpha(-x)\beta(s+x))dx\right)$$

where the contour C^+ contains the entire non-negative half complex plane and

$$\Phi(s,\omega)=\exp\left(\frac{1}{2\pi\sqrt{-1}\,s}\oint_{C^-}\log\left(\frac{x}{x-\omega}\right)\frac{\partial}{\partial x}\log(1-\alpha(s-x)\beta(x))dx\right)$$

where the contour C^- contains the entire negative half complex plane. The proof of this requires heavy machinery from complex analysis. Although this is a result in closed form, we do not believe that it is numerically useful, since the inversion of such a transform is numerically unstable due to the multiple branch points in the integrand.

In order to find numerically useful results, we consider two special cases, i.e. the $R/G/1$ and the $G/R/1$ queue (R is the class of distributions with rational Laplace transforms), when one of the distributions has a rational Laplace transform. We obtain closed-form solutions for these two cases which are computationally very tractable. As pointed out by a referee, an alternative approach is due to Asmussen [1], who converts a two-dimensional Hilbert factorization problem to a matrix-algorithmic form, which should then be solved numerically.

3. The solution of the Hilbert problem for the $R/G/1$ queue

In this case $\alpha(s) = \alpha_N(s)/\alpha_D(s)$, where $\alpha_D(s)$ is a monic polynomial in s of degree L and $\alpha_N(s)$ is a polynomial of degree less than L.

For fixed s with $\mathrm{Re}(s) \geqq 0$, let $z = x_r(s)$, $(r = 1, \cdots, L)$ be the L roots of the equation

$$\alpha(s-z)\beta(z) = 1, \qquad \mathrm{Re}(z) \geqq 0. \tag{13}$$

The proof of this follows along the lines of claim 3 of [2]. Once the number of roots is established through Rouché's theorem, we simply follow the methods pioneered by Keilson [6], [7]. Now, (9) can be written as

$$\frac{\Phi^+(s,\omega)}{\dfrac{\prod_{r=1}^L (x_r(s)-\omega)}{\alpha_D(s-\omega)-\alpha_N(s-\omega)\beta(\omega)}} = \frac{1-\rho^-(s,\omega)}{\dfrac{\prod_{r=1}^L (x_r(s)-\omega)}{\alpha_D(s-\omega)}}. \tag{14}$$

By observing that the expression in the left-hand side of Equation (14) is analytic for $\mathrm{Re}(\omega) > 0$ and the expression in the right-hand side of Equation (14) is analytic for $\mathrm{Re}(\omega) < 0$ and using Liouville's theorem we conclude that both expressions should be equal to a function of s. From the boundary conditions of (9) we easily find that the function is a constant function 1. To complete Liouville's theorem, we need the following proposition.

Proposition 1. *The expressions in both sides of Equation* (14) *are bounded.*

Proof. Let $\mathrm{Re}(s) \geqq 0$. For the left-hand side with $\mathrm{Re}(\omega) \geqq 0$, it is easily seen (since the zeros cancel out) that the denominator is bounded away from 0, and thus for some $\varepsilon > 0$

$$\left| \frac{\prod_{r=1}^L (x_r(s)-\omega)}{\alpha_D(s-\omega)-\alpha_N(s-\omega)\beta(\omega)} \right| \geqq \varepsilon.$$

We then check that the numerator is also bounded:

$$\begin{aligned} |\Phi^+(s,\omega)| &\leqq \Phi^+(0,0) \\ &= \int_0^\infty \int_0^\infty \sum_{n=0}^\infty f_n(x,y)\,dx\,dy \end{aligned}$$

$$\leqq \sum_{n=0}^{\infty} \left| \int_0^{\infty} \int_0^{\infty} f_n(x, y) dx\, dy \right|$$

$$\leqq 1 + \sum_{n=0}^{\infty} \Pr\left\{ \sum_{r=k+1}^{k+n} \xi_r > 0 \right\}.$$

Since $\rho < 1$, $E[\xi_r] < 0$ $(\forall r)$. As a result, applying the Chernoff bound, we obtain that there exists a constant $\delta < 1$ such that $\Pr\{\Sigma_{r=k+1}^{k+n} \xi_r > 0\} < \delta^n$, and thus

$$|\Phi^+(s, \omega)| \leqq \frac{1}{1-\delta} < \infty.$$

In an analogous way the denominator of the right-hand side with $\mathrm{Re}(\omega) \leqq 0$ is bounded away from 0, i.e., for some $\varepsilon > 0$:

$$\left| \frac{\Pi_{r=1}^{L} (x_r(s) - \omega)}{\alpha_{\mathrm{D}}(s - \omega)} \right| \geqq \varepsilon.$$

In addition the boundedness of the numerator of the right-hand side is seen as follows:

$$\begin{aligned} |1 - \rho^-(s, \omega)| &\leqq 1 + |\rho^-(s, \omega)| \\ &\leqq 1 + \rho^-(0, 0) = 1 + \sigma(0) = 2 \\ &< \infty. \end{aligned}$$

Thus by applying Liouville's theorem we conclude that the unique solution to the Hilbert factorization problem (9) is:

$$\Phi^+(s, \omega) = \frac{\Pi_{r=1}^{L} (x_r(s) - \omega)}{\alpha_{\mathrm{D}}(s - \omega) - \alpha_{\mathrm{N}}(s - \omega)\beta(\omega)}$$

$$\rho^-(s, \omega) = 1 - \frac{\Pi_{r=1}^{L} (x_r(s) - \omega)}{\alpha_{\mathrm{D}}(s - \omega)}.$$

Hence we get from (12)

$$\Phi(s, \omega) = \frac{\alpha_{\mathrm{D}}(s) - \alpha_{\mathrm{N}}(s)}{s(\alpha_{\mathrm{D}}(s - \omega) - \alpha_{\mathrm{N}}(s - \omega)\beta(\omega))} \prod_{r=1}^{L} \frac{x_r(s) - \omega}{x_r(s)} \tag{15}$$

and from (10)

$$\sigma(s) = 1 - \frac{1}{\alpha_{\mathrm{D}}(0)} \prod_{r=1}^{L} (x_r(s) - s). \tag{16}$$

The reward of our analysis is a simple closed-form expression for the transform of the busy period and waiting-time distribution. Moreover, we can find closed-form expressions for the first two moments of the waiting time and busy period distribution by differentiating the corresponding transforms. The following formulae were derived using the symbolic differentiation routine of the software package Mathematica on a Macintosh II computer:

$$\int_0^\infty e^{-sx} E[W^+ \mid \tau = x] dx = -\lim_{\omega \to 0} \frac{\partial}{\partial \omega} \Phi(s, \omega)$$

$$= \frac{1}{s} \left(\sum_{r=1}^{L} \frac{1}{x_r(s)} + \frac{(1/\mu)\alpha(s) + \dot{\alpha}(s)}{1 - \alpha(s)} - \frac{\dot{\alpha}_D(s)}{\alpha_D(s)} \right),$$

$$\int_0^\infty e^{-sx} E[(W^+)^2 \mid \tau = x] dx$$

$$= \lim_{\omega \to 0} \frac{\partial^2}{\partial \omega^2} \Phi(s, \omega)$$

$$= \frac{1}{s} \left(-\sum_{r=1}^{L} \frac{1}{x_r(s)^2} + \left(\frac{\dot{\alpha}_D(s)}{\alpha_D(s)} \right)^2 - \frac{\ddot{\alpha}_D(s)}{\alpha_D(s)} \right.$$

$$+ \frac{(1 + C_X^2)\alpha(s) - C_X^2 \alpha(s)^2 + 2\mu \dot{\alpha}(s) + \mu^2 \dot{\alpha}(s)^2 + \mu^2 \ddot{\alpha}(s) - \mu^2 \alpha(s) \ddot{\alpha}(s)}{\mu^2 (1 - \alpha(s))^2}$$

$$\left. + \left(\sum_{r=1}^{L} \frac{1}{x_r(s)} + \frac{(1/\mu)\alpha(s) + \dot{\alpha}(s)}{1 - \alpha(s)} - \frac{\dot{\alpha}_D(s)}{\alpha_D(s)} \right)^2 \right),$$

$$E[B_P] = -\lim_{s \to 0} \frac{d}{ds} \sigma(s) = \frac{\rho}{(1 - \rho)\alpha_D(0)} \prod_{r=1}^{L-1} x_r(0),$$

$$\mathrm{Var}[B_P] = \lim_{s \to 0} \frac{d^2}{ds^2} \log(\sigma(s))$$

$$= \frac{\rho}{(1 - \rho)\alpha_D(0)} \sum_{r=1}^{L-1} \frac{\alpha(-x_r(0))\dot{\beta}(x_r(0))}{\dot{\alpha}(-x_r(0))\beta(x_r(0)) - \alpha(-x_r(0))\dot{\beta}(x_r(0))} \prod_{\substack{k=1 \\ k \neq r}}^{L-1} x_k(0)$$

$$- \frac{\rho^2 (C_T^2 + C_X^2)}{(1 - \rho)^3 \lambda \alpha_D(0)} \prod_{r=1}^{L-1} x_r(0) - \left(\frac{\rho}{(1 - \rho)\alpha_D(0)} \right)^2 \prod_{r=1}^{L-1} x_r(0)^2,$$

where we used

$$\frac{d}{ds}(s - x_r(s)) = \frac{\alpha(s - x_r(s))\dot{\beta}(x_r(s))}{\alpha(s - x_r(s))\dot{\beta}(x_r(s)) - \dot{\alpha}(s - x_r(s))\beta(x_r(s))},$$

and

$$\frac{d^2}{ds^2}(s - x_r(s))$$

$$= \alpha(s - x_r(s))\beta(x_r(s))$$

$$\times [\alpha(s - x_r(s))\ddot{\alpha}(s - x_r(s))\dot{\beta}(x_r(s))^2 - 2\dot{\alpha}(s - x_r(s))^2 \dot{\beta}(x_r(s))^2$$

$$+ \dot{\alpha}(s - x_r(s))^2 \beta(x_r(s))\ddot{\beta}(x_r(s))]$$

$$\times [(\alpha(s - x_r(s))\dot{\beta}(x_r(s)) - \dot{\alpha}(s - x_r(s))\beta(x_r(s)))^3]^{-1}.$$

The formula for the first two moments of the busy period was simplified using the observation that there exists a unique root such that $x_L(0) = 0$ (see Keilson [8]). As an additional check of the algebra we can verify that for the $M/G/1$ queue, i.e. $L = 1$, the formula for $E[B_P]$ becomes $E[B_P] = 1/(\mu - \lambda)$. Finally, note that the roots $x_r(0)$ are precisely the roots that appear in the steady solution of the $R/G/1$ queue.

4. The solution of the Hilbert problem for the $GI/R/1$ queue

In this case $\beta(s) = \beta_N(s)/\beta_D(s)$, where $\beta_D(s)$ is a monic polynomial in s of degree M and $\beta_N(s)$ is a polynomial of degree less than M.

As in the previous section, for fixed s with $\mathrm{Re}(s) \geqq 0$, let $z = x_r(s)$ $(r = 1, \cdots, M)$ be the M roots of the equation

$$\alpha(s - z)\beta(z) = 1, \qquad \mathrm{Re}(z) < 0.$$

The unique solution to the Hilbert problem can be found in a similar way as in the previous section to be:

$$\Phi^+(s, \omega) = \frac{\beta_D(\omega)}{\Pi_{r=1}^{M} (\omega - x_r(s))}$$

$$\rho^-(s, \omega) = 1 - \frac{\beta_D(\omega) - \alpha(s - \omega)\beta_N(\omega)}{\Pi_{r=1}^{M} (\omega - x_r(s))}.$$

Note that the connection with the results of the previous section in the case of $R/R/1$ is established by noticing that

$$(-1)^L \prod_{r=1}^{L+M} (\omega - x_r(s)) = \alpha_D(s - \omega)\beta_D(\omega) - \alpha_N(s - \omega)\beta_N(\omega).$$

Hence we get from (10) and (12) that

$$\Phi(s, \omega) = \frac{\beta_D(\omega)}{s\beta_D(0)} \prod_{r=1}^{M} \frac{x_r(s)}{x_r(s) - \omega}, \tag{17}$$

$$\sigma(s) = 1 - \frac{\beta_D(s) - \beta_N(s)}{\Pi_{r=1}^{M} (s - x_r(s))}. \tag{18}$$

As an accuracy check we can easily check that (17) and (18) are identical with the results for the $MGE_L/MGE_M/1$ queue obtained in Bertsimas and Nakazato [2]. As in the previous section we can find closed-form formulae for the moments of the distributions involved as follows:

$$\int_0^\infty e^{-sx} E[W^+ \mid \tau = x] dx = -\frac{1}{s}\left(\sum_{r=1}^{M} \frac{1}{x_r(s)} + \frac{\dot\beta_D(0)}{\beta_D(0)}\right),$$

$$\int_0^\infty e^{-sx}E[(W^+)^2 \mid \tau = x]dx$$

$$= \frac{1}{s}\left(\sum_{r=1}^{M} \frac{1}{x_r(s)^2} + \frac{\ddot{\beta}_D(0)}{\beta_D(0)} - \left(\frac{\dot{\beta}_D(0)}{\beta_D(0)}\right)^2 + \left(\sum_{r=1}^{M} \frac{1}{x_r(s)} + \frac{\dot{\beta}_D(0)}{\beta_D(0)}\right)^2\right),$$

$$E[B_P] = \frac{(-1)^M \beta_D(0)}{\mu} \prod_{r=1}^{M} \frac{1}{x_r(0)},$$

$$\operatorname{Var}[B_P] = \left(\frac{(-1)^M \beta_D(0)}{\mu} \sum_{k=1}^{M} \frac{\alpha(-x_k(0))\dot{\beta}(x_k(0))}{x_k(0)\{\dot{\alpha}(-x_k(0))\beta(x_k(0)) - \alpha(-x_k(0))\dot{\beta}(x_k(0))\}}\right.$$

$$\left. + \frac{(-1)^M(1 + C_X^2)\beta_D(0)}{\mu^2} - \frac{\dot{\beta}_D(0)}{\mu}\right) \prod_{r=1}^{M} \frac{1}{x_r(0)} - \left(\frac{\beta_D(0)}{\mu}\right)^2 \prod_{r=1}^{M} \frac{1}{x_r(0)^2}.$$

5. The $M/G/1$ and $GI/M/1$ queues

In this section we verify and generalize well-known results for the $GI/M/1$ and $M/G/1$ queues.

For $GI/M/1$ it is known (Takács [12]) that $\sigma(s) = \mu(1 - w(s))/(s + \mu - w(s))$, where $w(s) = a(s - x_1(s))$. By letting $M = 1$ in (18) and observing that $w(s)\beta(x_1(s)) = 1$, i.e. $x_1(s) = \mu(w(s) - 1)$, we find the same expression.

For the $M/G/1$ queue it is well known (see Kleinrock [9]) that the busy period satisfies $\sigma(s) = \beta(s + \lambda - \lambda\sigma(s))$. In order to see how we can derive this from (16) we observe that from (16) $\sigma(s) = 1 + (s - x_1(s))/\lambda$, whence $x_1(s) = s + \lambda - \lambda\sigma(s)$. Since $x_1(s)$ satisfies from (13)

$$\alpha(s - x_1(s))\beta(x_1(s)) = \frac{\lambda}{\lambda + s - x_1(s)} \beta(s + \lambda - \lambda\sigma(s)) = 1,$$

we can now easily derive the desired relation $\sigma(s) = \beta(s + \lambda - \lambda\sigma(s))$.

The time-dependent behavior of the waiting time can be expressed in terms of $\sigma(s)$ as follows:

$$\Phi(s, \omega) = \frac{1}{s + \lambda(1 - \sigma(s))} \frac{s - \omega + \lambda(1 - \sigma(s))}{s - \omega + \lambda(1 - \beta(\omega))}.$$

This is a solution to the well-known Takács integrodifferential equation (see Kleinrock [9] or Takács [13]).

6. Concluding remarks

In this paper we attempted to demonstrate the power of direct probabilistic arguments for the waiting-time distribution in the transient domain and the busy period distribution for the $GI/G/1$ queue. We found closed-form expressions for the transforms and the first two moments of these distributions. Algorithmically our approach offers a method for finding these distributions in the time domain through the numerical inversion of the Laplace transforms. In Bertsimas and Nakazato [2] we reported numerical results for finding numerically the busy period, the transient queue length and the waiting time distributions in an $MGE/MGE/1$ queue, by numerically inverting the corresponding Laplace transforms.

References

[1] ASMUSSEN, S. (1989) Aspects of matrix Wiener–Hopf factorisation in applied probability. *Math. Sci.* **14**, 101–116.

[2] BERTSIMAS, D. AND NAKAZATO, D. (1989) Transient and busy period analysis of the *GI/G/1* queue using the method of stages. *QUESTA*. To appear.

[3] GRASSMANN, W. K. (1977) Transient solutions in Markovian queueing systems. *Comput. Operat Res.* **4**, 47–53.

[4] GROSS, D. AND HARRIS, C. (1985) *Fundamentals of Queueing Theory*. Wiley, New York.

[5] JENSEN, A. (1953) Markoff chains as an aid in the study of Markoff processes. *Skand. Aktuarietidskr* **36**, 87–91.

[6] KEILSON, J. (1961) The homogeneous random walk on the half-line and the Hilbert problem. *Bull. Inst. Internat. Statist.* 33rd Session, Paper 113, 1–13.

[7] KEILSON, J. (1962) General bulk queue as a Hilbert problem. *J. R. Statist. Soc.* **24**, 344–358.

[8] KEILSON, J. (1969) On the matrix renewal function for Markov renewal process. *Ann. Math. Statist.* **40**, 1901–1907.

[9] KLEINROCK, L. (1975) *Queueing Systems*; *Vol.* 1: *Theory*. Wiley, New York.

[10] LINDLEY, D. (1952) The theory of queues with a single server. *Proc. Camb. Phil. Soc.* **48**, 277–289.

[11] RAMASWAMI, V. (1982) The busy period of queues which have a matrix-geometric steady state probability vector. *Opsearch* **19**, 238–261.

[12] TAKÁCS, L. (1962) *Introduction to the Theory of Queues*. Oxford University Press, New York.

[13] TAKÁCS, L. (1962) A single-server queue with Poisson input. *Operat. Res.* **10**, 388–397.